Consumer Surplus Revisited

Daniel Schaffa*
University of Richmond
School of Law

November 3, 2018

Economists have long studied how changes in prices affect consumer wellbeing. Despite the considerable progress made towards resolving this fundamental economic inquiry, central questions remain. This paper revisits a topic that has not received much attention in recent years: consumer surplus.

Dupuit (1844) is credited with first noting that demand functions could be used infer the changes in consumer welfare due to price changes.[1] It did not take long for critics to surface. Bordas (1847) noted that consumer surplus as Dupuit described it could not account for changes in either income or other prices. Thus the question of how to think about consumer surplus in a multi-good setting was raised for the first time.

Hotelling (1938) was the first to suggest that "the natural generalization [to n commodities] of the integral representing total benefit, of which consumers' surplus is a part, is [a] line integral." Hotelling also noted that the line integral depended on the path of integration. Auerbach (1985) provides the most familiar formulation of the argument that the change in consumer surplus from one price vector to another will depend on the path through price space that this change takes.

This, however, is inconsistent with the definition of consumer surplus as the area under the demand curve from the current price to an infinitely high price. The area under a curve is a mathematical object that depends only on the the curve and the bounds of integration. The change in consumer surplus is then the difference between two areas. Because both areas are functions of a price vector and not a price path, the difference between these areas must also be independent of path.

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*dschaffa@gmail.com. I thank Jim Hines, Dan Jaqua, and attendees of the University of Michigan Public Finance Seminar for helpful comments. I gratefully acknowledge support from the NIA training grant to the Population Studies Center at the University of Michigan (T32 AG000221). Any errors are my own.

[1] See Ekelund and Hebert (1985); Hines (1999) for further historical detail on the development of consumer surplus.

While the area under the demand curve does not suffer from path-dependence, it is nonetheless imperfect. For some utility functions, including CES with $r \leq 0$, the change in consumer surplus is an adequate stand-in for change in utility, but there are also cases in which change in consumer surplus and change in utility have different signs. Consumer surplus as the area under a curve also raises conceptual issues. In many cases consumer surplus is infinite. In some cases the change in consumer surplus is infinite. For several utility functions, increasing all prices and income by the same factor can change consumer surplus. Most fundamentally, consumer surplus does not lend itself to an obvious economic interpretation.

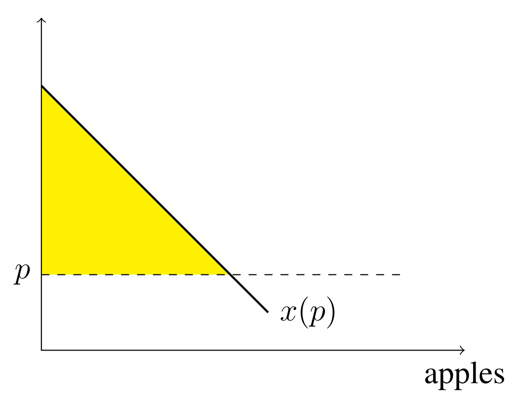
While consumer surplus may have little use as an analytical tool, it still bears correcting the misconception that the area under the curve can be path dependent.

1 Consumer surplus as the area under a curve

Consumer surplus is defined as the area under the demand curve from a given price to a price of infinity. In one dimension:

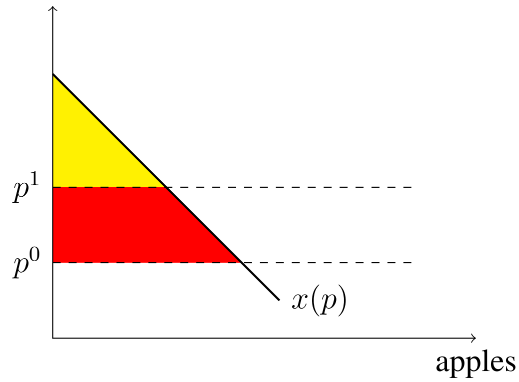
$$\text{CS}(p) = \int_p^\infty x(t)dt \tag{1}$$

Which can be graphically represented.



If the price of the good changes, then the consumer surplus changes. If, say, the price of apples increases, the new consumer surplus is the yellow (lighter grey) area and the change in consumer surplus is the red (darker gray area).[2]

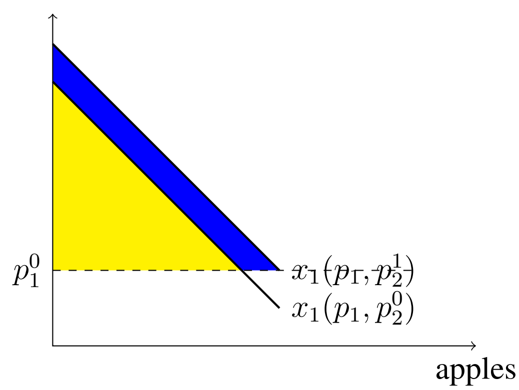
[2] I use subscripts to index goods and superscripts to index price vectors.



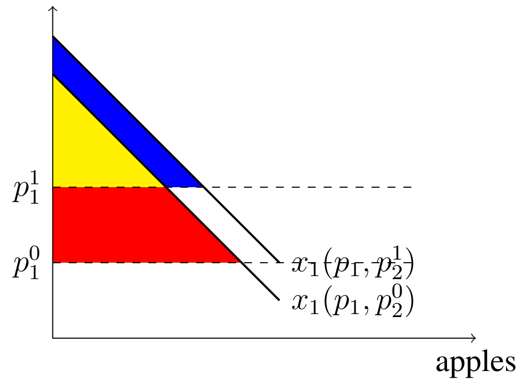
In the single good case, the change in consumer surplus can be rewritten as a single integral, but this does not generalize to 2 or more dimensions.

$$\Delta \text{CS} = \text{CS}(p^1) - \text{CS}(p^0) = \int_{p^1}^{\infty} x(t)dt - \int_{p^0}^{\infty} x(t)dt = \int_{p^1}^{p^0} x(t)dt \tag{2}$$

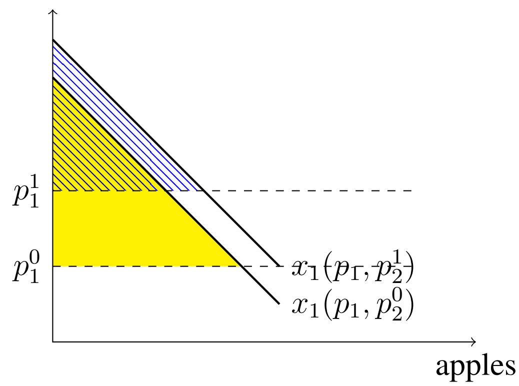
A line integral in 2 or more dimensions is not a change. To understand how consumer surplus may be computed with arbitrary goods, consider first the two-good case. Just as with the single-good case, consumer surplus changes with own price changes, but consumer surplus also changes with cross price changes. As cross-price changes shift the demand curve, consumer surplus changes.



The blue (dark grey) area is the increase in consumer surplus. Of course it is possible for both own- and cross-price changes to take place simultaneously.



Note that the order of the price change does not affect either the initial or final consumer surplus.



CS_1^0 (yellow) and CS_1^1 (blue stripped) are the same regardless of which price changes first; ΔCS is same regardless of path. If p_1 is changed first, then the parallelogram formed between the two demand curves is not included in the ΔCS for either good. If p_2 is changed first, then the parallelogram formed between the two demand curves is included in the ΔCS for both goods and is thereby cancelled out.

The initial consumer surplus from x_1 is

$$\mathrm{CS}_1^0 = \int_{p_1^0}^{\infty} x_1(t_1, p_2^0) dt_1$$

After the price change, the consumer surplus from x_1 is

$$\mathrm{CS}_1^1 = \int_{p_1^1}^{\infty} x_1(t_1, p_2^1) dt_1$$

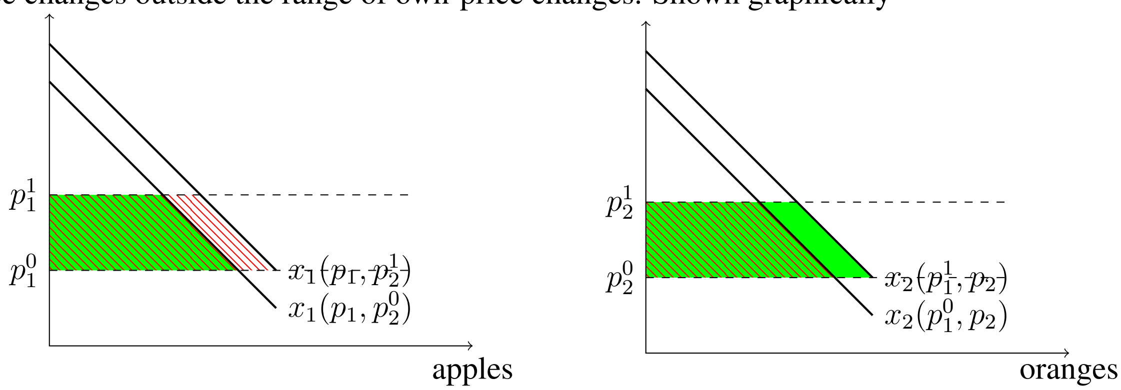
The change in consumer surplus from x_1 is

$$\Delta \text{CS}_1 = \text{CS}_1^1 - \text{CS}_1^0 = \int_{p_1^1}^{\infty} x_1(t_1, p_2^1) dt_1 - \int_{p_1^0}^{\infty} x_1(t_1, p_2^0) dt_1$$

which cannot be represented as a single integral bounded by p_1^0 and p_1^1 because p_2 has also changed. Previous work, summarized in Auerbach (1985), fails to account for the cross-price effect on the change in consumer surplus. It does show that

$$\int_{p_1^0}^{p_1^1} x_1(p_1, p_2^0) dp_1 + \int_{p_2^0}^{p_2^1} x_2(p_1^1, p_2) dp_2 \neq \int_{p_2^0}^{p_2^1} x_2(p_1^0, p_2) dp_2 + \int_{p_1^0}^{p_1^1} x_1(p_1, p_2^1) dp_1$$

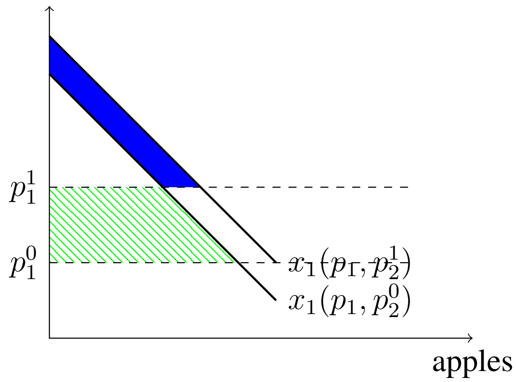
However, these sets of integrals fail to account for changes in consumer surplus due to cross-price changes outside the range of own-price changes. Shown graphically

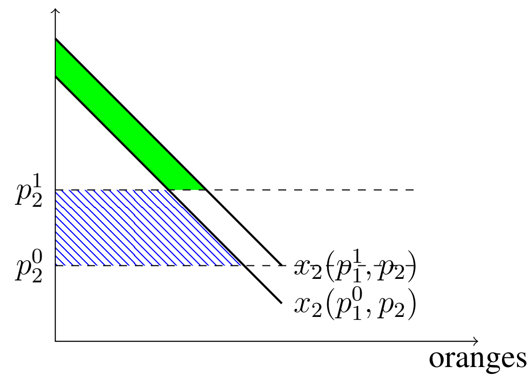


$$\text{green: } \int_{p_1^0}^{p_1^1} x_1(p_1, p_2^0) dp_1 + \int_{p_2^0}^{p_2^1} x_2(p_1^1, p_2) dp_2 \neq$$

$$\text{stripped red: } \int_{p_2^0}^{p_2^1} x_2(p_1^0, p_2) dp_2 + \int_{p_1^0}^{p_1^1} x_1(p_1, p_2^1) dp_1$$

When the cross price effect is included, the order of the price change does not matter, as seen below.





The solid areas are increases, and the striped areas are decreases. The green areas are changes induced by the the price of apples, and the blue areas are changes induced by the price of oranges. These areas will be the same regardless of the order, or path, of price changes.

$$\Delta \text{CS}_i = \int_{p_i^1}^{\infty} x_i(p_1^1, ..., p_i, ..., p_n^1) dp_i - \int_{p_i^0}^{\infty} x_i(p_1^0, ..., p_i, ..., p_n^0) dp_i =$$

$$\int_{p_i^1}^{\infty} x_i(p_1^1, ..., p_i, ..., p_n^1) dp_i - \int_{p_i^1}^{\infty} x_i(p_1^0, ..., p_i, ..., p_n^0) dp_i$$

$$- \int_{p_i^0}^{p_i^1} x_i(p_1^0, ..., p_i, ..., p_n^0) dp_i =$$

$$\underbrace{\int_{p_i^1}^{\infty} \left[x_i(p_1^1, ..., p_i, ..., p_n^1) - x_i(p_1^0, ..., p_i, ..., p_n^0) \right] dp_i}_{\text{surplus change from cross-price change}}$$

$$- \underbrace{\int_{p_i^0}^{p_i^1} x_i(p_1^0, ..., p_i, ..., p_n^0) dp_i}_{\text{surplus change from own-price change}}$$

Summing over all price changes

$$\Delta \text{CS} = \sum_i \Delta \text{CS}_i = \sum_i \left[\int_{p_i^1}^{\infty} x_i(p_1^1, ..., p_i, ..., p_n^1) dp_i - \int_{p_i^0}^{\infty} x_i(p_1^0, ..., p_i, ..., p_n^0) dp_i \right]$$

It is worth noting that the decomposition of the change in surplus is path dependent even though the total is not.

$$\Delta\text{CS}_i = \underbrace{\int_{p_i^1}^{\infty} \left[x_i(p_1^1, ..., p_i, ..., p_n^1) - x_i(p_1^0, ..., p_i, ..., p_n^0)\right] dp_i}_{\text{surplus change from cross-price change, own-price first}}$$

$$- \underbrace{\int_{p_i^0}^{p_i^1} x_i(p_1^0, ..., p_i, ..., p_n^0) dp_i}_{\text{surplus change from own-price change, own-price first}}$$

$$= \underbrace{\int_{p_i^0}^{\infty} \left[x_i(p_1^1, ..., p_i, ..., p_n^1) - x_i(p_1^0, ..., p_i, ..., p_n^0)\right] dp_i}_{\text{surplus change from cross-price change, cross-price first}}$$

$$- \underbrace{\int_{p_i^0}^{p_i^1} x_i(p_1^1, ..., p_i, ..., p_n^1) dp_i}_{\text{surplus change from own-price change, cross-price first}}$$

Because

$$\int_{p_1^0}^{p_1^1} x_1(p_1, p_2^0) dp_1 + \int_{p_2^0}^{p_2^1} x_2(p_1^1, p_2) dp_2$$

$$\neq \int_{p_2^0}^{p_2^1} x_2(p_1^0, p_2) dp_2 + \int_{p_1^0}^{p_1^1} x_1(p_1, p_2^1) dp_1$$

2 The welfare properties of consumer surplus

That the change in consumer surplus is not path dependent makes it a likelier candidate for use in welfare analysis. Ideally economists would observe indirect utility functions, in which case they would be able to know the precise welfare implications of any change in price. However, utility functions and not observable, so economists make assumptions about the underlying utility functions to draw welfare conclusions from quantity and price data.

If changes in consumer surplus are additive[3] and always have the same sign as changes in indirect utility, then the change in consumer will relay all the relevant information about welfare.

Consumer surplus has the advantage of requiring only an estimate of the demand system to be computable, and the change in consumer surplus is always additive. However, for many utility

[3]Meaning that the change in consumer surplus from price vector p to price vector q is equal to the change in consumer surplus from p to r added to the change in consumer surplus from r to q, i.e. $\Delta\text{CS}(p, q) = \Delta\text{CS}(p, r) + \Delta\text{CS}(r, q)$

functions, the change in consumer surplus does not have a sufficiently clear relationship to the change in indirect utility to be an adequate substitute. An exception is CES utility with $r \leq 0$.

$$u(x) = \frac{1}{r} \ln \left(\sum_i \alpha_i x_i^r \right)$$

For that utility function, the change in indirect utility caused by moving from price p to p' is[4]

$$\Delta v = \frac{1-r}{r} \ln \left[\frac{\sum_i \alpha_i^{\frac{-1}{r-1}} p_i'^{\frac{r}{r-1}}}{\sum_i \alpha_i^{\frac{-1}{r-1}} p_i^{\frac{r}{r-1}}} \right]$$

The change in consumer surplus caused by moving from price p to p' is

$$\Delta CS = \lim_{z \to \infty} \frac{y(r-1)}{r} \sum_i \left[\ln \left(\frac{z^{\frac{r}{r-1}} \alpha_i^{\frac{-1}{r-1}} + \sum_{j \neq i} p_j'^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}{\sum_j p_j'^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}} \frac{\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}{z^{\frac{r}{r-1}} \alpha_i^{\frac{-1}{r-1}} + \sum_{j \neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}} \right) \right]$$

For all CES utility functions with weakly complementary goods, $r \leq 0$, taking the limit yeilds

$$\Delta CS = \frac{ny(1-r)}{r} \ln \left(\frac{\sum_j p_j'^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}{\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}} \right)$$

Thus $\Delta CS = ny \Delta v$, meaning that computing the change in consumer surplus is sufficient to make welfare claims. However, when goods are substitutable, the change in consumer surplus may have the opposite sign as the change in indirect utility. Consider two perfect substitutes with prices such that the consumer is indifferent between between them. The area under the both demand curves is 0 because any increase in price leads to 0 quantity demanded. Thus consumer surplus is 0 for both goods. If either price increases, then there will be positive consumer surplus for the cross-good. However, indirect utility will decrease because the consumer can only afford a less preferred bundle at the higher price.

Other problems arise for several utility functions. For Stone-Geary utility, $u(x) = \sum_i \beta_i \ln (x_i - \gamma_i)$, consumer surplus is

$$CS(p) = \lim_{z \to \infty} \sum_i \left[(1-\beta_i)\gamma_i z + (p_i x_i - (1-\beta_i) p_i \gamma_i) \ln z - (1-\beta_i)\gamma_i p_i - (p_i x_i - (1-\beta_i) p_i \gamma_i) \ln p_i \right]$$

[4] All derivations in appendix.

which is unbounded. Even more problematic, the change in consumer surplus will be unbounded for price changes such that $\sum_i (1-\beta_i)(p_i\gamma_i - p_i'\gamma_i) \neq 0$.

$$\Delta CS =$$

$$\lim_{z\to\infty} \sum_i (1-\beta_i)(p_i\gamma_i - p_i'\gamma_i)\ln z - (1-\beta_i)(\gamma_i p_i' - \gamma_i p_i) - (p_i'x_i' - (1-\beta_i)p_i'\gamma_i)\ln p_i' + (p_i x_i - (1-\beta_i)p_i\gamma_i)\ln p_i$$

3 Conclusion

Because consumer surplus is only useful as a means to measure the change in welfare for some utility functions, it has little use as an analytical tool Indeed, making the assumption that consumer surplus will be an adequate surrogate for indirect utility function essentially assumes the utility function. Doing so makes direct computation of the change in utility preferred to computing consumer surplus.

Nonetheless, if we agree as a discipline that consumer surplus is the area under a curve, then the change in consumer surplus will not be path dependent. To the extent that our articles, books, and lesson plans state otherwise, we should correct them.

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A n good Constant elasticity

$$u(x) = \left(\sum_i \alpha_i x_i^r \right)^{\frac{1}{r}}$$

$$u(x) = \frac{1}{r} \ln \left(\sum_i \alpha_i x_i^r \right)$$

such that

$$\sum_i p_i x_i = y$$

$$\mathcal{L} = \frac{1}{r} \ln \left(\sum_i \alpha_i x_i^r \right) + \lambda \left(y - \sum_i p_i x_i \right)$$

FOC

$$\frac{1}{r}\frac{r\alpha_i x_i^{r-1}}{\sum_k \alpha_k x_k^r} = \lambda p_i$$

$$\alpha_i x_i^{r-1} = \lambda p_i \sum_k \alpha_k x_k^r$$

$$\alpha_i x_i^{r-1}\left(\lambda p_j \sum_k \alpha_k x_k^r\right) = \alpha_j x_j^{r-1}\left(\lambda p_i \sum_k \alpha_k x_k^r\right)$$

$$\alpha_i x_i^{r-1} p_j = \alpha_j x_j^{r-1} p_i$$

$$\alpha_i^{\frac{1}{r-1}} x_i p_j^{\frac{1}{r-1}} p_i^{\frac{-1}{r-1}} \alpha_j^{\frac{-1}{r-1}} = x_j$$

$$\sum_j p_j \alpha_i^{\frac{1}{r-1}} x_i p_j^{\frac{1}{r-1}} p_i^{\frac{-1}{r-1}} \alpha_j^{\frac{-1}{r-1}} = \sum_j p_j x_j$$

$$\alpha_i^{\frac{1}{r-1}} x_i p_i^{\frac{-1}{r-1}} \sum_j p_j^{\frac{1+r-1}{r-1}} \alpha_j^{\frac{-1}{r-1}} = y$$

$$x_i = \frac{\alpha_i^{\frac{-1}{r-1}} p_i^{\frac{1}{r-1}} y}{\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}$$

$$v(p) = \frac{1}{r}\ln\left[\sum_i \alpha_i \left(\frac{\alpha_i^{\frac{-1}{r-1}} p_i^{\frac{1}{r-1}} y}{\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}\right)^r\right]$$

$$v(p) = \frac{1}{r}\ln\left[\left(\frac{y}{\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}\right)^r \sum_i \alpha_i \left(\alpha_i^{\frac{-1}{r-1}} p_i^{\frac{1}{r-1}}\right)^r\right]$$

$$v(p) = \frac{1}{r}\ln\left(\frac{y}{\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}\right)^r + \frac{1}{r}\ln\left[\sum_i \alpha_i \left(\alpha_i^{\frac{-r}{r-1}} p_i^{\frac{r}{r-1}}\right)\right]$$

$$v(p) = \ln y - \ln\left(\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}\right) + \frac{1}{r}\ln\left[\sum_i \alpha_i^{\frac{-1}{r-1}} p_i^{\frac{r}{r-1}}\right]$$

$$v(p) = \ln y + \frac{1-r}{r}\ln\left[\sum_i \alpha_i^{\frac{-1}{r-1}} p_i^{\frac{r}{r-1}}\right]$$

$$\Delta v = \frac{1-r}{r} \ln \left[\sum_i \alpha_i^{\frac{-1}{r-1}} {p'}_i^{\frac{r}{r-1}} \right] - \frac{1-r}{r} \ln \left[\sum_i \alpha_i^{\frac{-1}{r-1}} p_i^{\frac{r}{r-1}} \right]$$

$$\Delta v = \frac{1-r}{r} \ln \left[\frac{\sum_i \alpha_i^{\frac{-1}{r-1}} {p'}_i^{\frac{r}{r-1}}}{\sum_i \alpha_i^{\frac{-1}{r-1}} p_i^{\frac{r}{r-1}}} \right]$$

$$CS = \sum_i \left[y \int_{p_i}^{z_i} \left(\frac{\alpha_i^{\frac{-1}{r-1}} t_i^{\frac{1}{r-1}}}{t_i^{\frac{r}{r-1}} \alpha_i^{\frac{-1}{r-1}} + \sum_{j \neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}} \right) dt_i \right]$$

$$CS = \sum_i \left[y \int_{p_i}^{z_i} \left(\frac{\frac{r-1}{r} \alpha_i^{\frac{-1}{r-1}} t_i^{\frac{1}{r-1}}}{\frac{r-1}{r} t_i^{\frac{r}{r-1}} \alpha_i^{\frac{-1}{r-1}} + \frac{r-1}{r} \sum_{j \neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}} \right) dt_i \right]$$

$$\int \frac{x^b}{x^{b+1} + y} = \frac{1}{b+1} \int \frac{x^b}{x^{b+1}/(b+1) + y/(b+1)} = \frac{1}{b+1} \ln(x^{b+1}/(b+1) + y/(b+1))$$

$$\int_{p_i}^{z_i} \left(\frac{\alpha_i^{\frac{-1}{r-1}} t_i^{\frac{1}{r-1}}}{t_i^{\frac{r}{r-1}} \alpha_i^{\frac{-1}{r-1}} + \sum_{j \neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}} \right) dt_i$$

$$\frac{r-1}{r} \int_{p_i}^{z_i} \left(\frac{\alpha_i^{\frac{-1}{r-1}} t_i^{\frac{1}{r-1}}}{\frac{r-1}{r} t_i^{\frac{r}{r-1}} \alpha_i^{\frac{-1}{r-1}} + \frac{r-1}{r} \sum_{j \neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}} \right) dt_i$$

$$= \frac{r-1}{r} \ln \left(\frac{r-1}{r} t_i^{\frac{r}{r-1}} \alpha_i^{\frac{-1}{r-1}} + \frac{r-1}{r} \sum_{j \neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}} \right) \Bigg|_{p_i}^{z_i}$$

$$CS = \frac{y(r-1)}{r}$$

$$\sum_i \left[\ln \left(\frac{r-1}{r} z_i^{\frac{r}{r-1}} \alpha_i^{\frac{-1}{r-1}} + \frac{r-1}{r} \sum_{j \neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}} \right) - \ln \left(\frac{r-1}{r} p_i^{\frac{r}{r-1}} \alpha_i^{\frac{-1}{r-1}} + \frac{r-1}{r} \sum_{j \neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}} \right) \right]$$

$$CS = \frac{y(r-1)}{r} \sum_i \left[\ln \left(\frac{\frac{r-1}{r} z_i^{\frac{r}{r-1}} \alpha_i^{\frac{-1}{r-1}} + \frac{r-1}{r} \sum_{j \neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}{\frac{r-1}{r} p_i^{\frac{r}{r-1}} \alpha_i^{\frac{-1}{r-1}} + \frac{r-1}{r} \sum_{j \neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}} \right) \right]$$

$$CS = \frac{y(r-1)}{r} \sum_i \left[\ln \left(\frac{z_i^{\frac{r}{r-1}} \alpha_i^{\frac{-1}{r-1}} + \sum_{j \neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}{\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}} \right) \right]$$

$$\Delta CS = \frac{y(r-1)}{r}\sum_i \left[\ln\left(\frac{z_i^{\frac{r}{r-1}}\alpha_i^{\frac{-1}{r-1}} + \sum_{j\neq i} p'^{\frac{r}{r-1}}_j \alpha_j^{\frac{-1}{r-1}}}{\sum_j p'^{\frac{r}{r-1}}_j \alpha_j^{\frac{-1}{r-1}}}\right) - \ln\left(\frac{z_i^{\frac{r}{r-1}}\alpha_i^{\frac{-1}{r-1}} + \sum_{j\neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}{\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}\right)\right]$$

$$\Delta CS = \frac{y(r-1)}{r}\sum_i \left[\ln\left(\frac{z_i^{\frac{r}{r-1}}\alpha_i^{\frac{-1}{r-1}} + \sum_{j\neq i} p'^{\frac{r}{r-1}}_j \alpha_j^{\frac{-1}{r-1}}}{\sum_j p'^{\frac{r}{r-1}}_j \alpha_j^{\frac{-1}{r-1}}} \frac{\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}{z_i^{\frac{r}{r-1}}\alpha_i^{\frac{-1}{r-1}} + \sum_{j\neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}\right)\right]$$

Note that $r \to -\infty$ is perfect complements; $r \to 0$ is Cobb-Douglas, and $r \to 1$ is perfect substitutes. $r > 1$ or $r < 0$ implies that $\frac{r}{r-1} > 0$, so

$$\Delta CS = \frac{y(r-1)}{r}\sum_i \left[\ln\left(\frac{\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}{\sum_j p'^{\frac{r}{r-1}}_j \alpha_j^{\frac{-1}{r-1}}}\right)\right]$$

$$\Delta CS = \frac{ny(r-1)}{r}\ln\left(\frac{\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}{\sum_j p'^{\frac{r}{r-1}}_j \alpha_j^{\frac{-1}{r-1}}}\right)$$

$$\Delta CS = \frac{ny(1-r)}{r}\ln\left(\frac{\sum_j p'^{\frac{r}{r-1}}_j \alpha_j^{\frac{-1}{r-1}}}{\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}\right)$$

which will have the same sign as

$$\Delta v = \frac{1-r}{r}\ln\left[\frac{\sum_i \alpha_i^{\frac{-1}{r-1}} p'^{\frac{r}{r-1}}_i}{\sum_i \alpha_i^{\frac{-1}{r-1}} p_i^{\frac{r}{r-1}}}\right]$$

$r \in (0,1) \implies \frac{r}{r-1} < 0$, so

$$\Delta CS = \frac{y(r-1)}{r}\sum_i \left[\ln\left(\frac{z_i^{\frac{r}{r-1}}\alpha_i^{\frac{-1}{r-1}} + \sum_{j\neq i} p'^{\frac{r}{r-1}}_j \alpha_j^{\frac{-1}{r-1}}}{\sum_j p'^{\frac{r}{r-1}}_j \alpha_j^{\frac{-1}{r-1}}} \frac{\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}{z_i^{\frac{r}{r-1}}\alpha_i^{\frac{-1}{r-1}} + \sum_{j\neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}\right)\right]$$

$$\Delta CS = \frac{y(r-1)}{r}\sum_i \left[\ln\left(\frac{\sum_{j\neq i} p'^{\frac{r}{r-1}}_j \alpha_j^{\frac{-1}{r-1}}}{\sum_j p'^{\frac{r}{r-1}}_j \alpha_j^{\frac{-1}{r-1}}} \frac{\sum_j p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}{\sum_{j\neq i} p_j^{\frac{r}{r-1}} \alpha_j^{\frac{-1}{r-1}}}\right)\right]$$

$$\Delta CS = \frac{y(1-r)}{r}\sum_i \left[\ln\left(\frac{\sum_j {p'}_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}{\sum_j p_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}\right) + \ln\left(\frac{\sum_{j\neq i} {p'}_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}{\sum_{j\neq i} p_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}\right)\right]$$

$$\Delta CS = \frac{ny(1-r)}{r}\ln\left(\frac{\sum_j {p'}_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}{\sum_j p_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}\right) + \frac{y(1-r)}{r}\sum_i \ln\left(\frac{\sum_{j\neq i} {p'}_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}{\sum_{j\neq i} p_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}\right)$$

$$\Delta CS = \frac{y(r-1)}{r}\sum_i \left[\ln\left(\frac{\sum_{j\neq i} {p'}_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}{\sum_j {p'}_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}\right)\left(\frac{\sum_j p_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}{\sum_{j\neq i} p_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}\right)\right]$$

$$\Delta CS = \frac{y(r-1)}{r}\sum_i \left[\ln\left(\frac{\sum_{j\neq i} {p'}_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}{\sum_j {p'}_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}\right)\left(\frac{\sum_j p_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}{\sum_{j\neq i} p_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}\right)\right]$$

$$\Delta CS = \frac{y(r-1)}{r}\sum_i \left[\ln\left(\frac{-{p'}_i^{\frac{r}{r-1}}\alpha_i^{\frac{-1}{r-1}} + \sum_j {p'}_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}{\sum_j {p'}_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}\right) - \ln\left(\frac{-p_i^{\frac{r}{r-1}}\alpha_i^{\frac{-1}{r-1}} + \sum_j p_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}{\sum_j p_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}\right)\right]$$

$$\Delta CS = \frac{y(r-1)}{r}\sum_i \left[\ln\left(1 - \frac{{p'}_i^{\frac{r}{r-1}}\alpha_i^{\frac{-1}{r-1}}}{\sum_j {p'}_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}\right) - \ln\left(1 - \frac{p_i^{\frac{r}{r-1}}\alpha_i^{\frac{-1}{r-1}}}{\sum_j p_j^{\frac{r}{r-1}}\alpha_j^{\frac{-1}{r-1}}}\right)\right]$$

B n good Cobb-Douglas

$$u(x) = \sum_i \alpha_i \ln x_i$$

such that

$$\sum_i p_i x_i = y$$

$$\mathcal{L} = \sum_i \alpha_i \ln x_i + \lambda(y - \sum_i p_i x_i)$$

FOC

$$\frac{\alpha_i}{x_i} = \lambda p_i$$

Thus

$$\lambda p_i \frac{\alpha_j}{x_j} = \lambda p_j \frac{\alpha_i}{x_i}$$

$$\sum_j p_i x_i \alpha_j = \sum_j p_j x_j \alpha_i$$

$$p_i x_i = \alpha_i y$$

$$x_i = \frac{\alpha_i y}{p_i}$$

check

$$\sum_i p_i \frac{\alpha_i y}{p_i} = \sum_i \alpha_i y = y$$

$$v(p) = \sum_i \alpha_i \ln\left(\frac{\alpha_i y}{p_i}\right) = \sum_i \alpha_i \left(\ln \alpha_i + \ln y - \ln p_i\right)$$

$$\Delta v = \sum_i \alpha_i \left(\ln \alpha_i + \ln y - \ln {p'}_i\right) - \sum_i \alpha_i \left(\ln \alpha_i + \ln y - \ln p_i\right)$$

$$\Delta v = -\sum_i \alpha_i \Delta \ln p_i$$

$$CS = \sum_i \int_{p_i}^{z_i} \frac{\alpha_i y}{t_i} dt_i$$

$$CS = y \sum_i \alpha_i (\ln z_i - \ln p_i)$$

$$\Delta CS = y \sum_i \alpha_i (\ln z_i - \ln {p'}_i) - y \sum_i \alpha_i (\ln z_i - \ln p_i)$$

$$\Delta CS = -y \sum_i \alpha_i \Delta \ln p_i$$

C n good Stone-Geary

$$u(x) = \prod_i (x_i - \gamma_i)^{\beta_i}$$

OR

$$u(x) = \sum_i \beta_i \ln(x_i - \gamma_i)$$

such that

$$\sum_i p_i x_i = y$$

Note: this assumes that all goods are positively consumed.

$$\mathcal{L} = \sum_i \beta_i \ln(x_i - \gamma_i) + \lambda \left(y - \sum_i p_i x_i \right)$$

FOC

$$\frac{\beta_i}{x_i - \gamma_i} = \lambda p_i$$

$$\lambda p_j \frac{\beta_i}{x_i - \gamma_i} = \lambda p_i \frac{\beta_j}{x_j - \gamma_j}$$

$$p_j \beta_i (x_j - \gamma_j) = p_i \beta_j (x_i - \gamma_i)$$

$$\sum_j p_j \beta_i x_j - \sum_j p_j \beta_i \gamma_j = \sum_j p_i \beta_j x_i - \sum_j p_i \beta_j \gamma_i$$

$$\beta_i \sum_j p_j x_j - \beta_i \sum_j p_j \gamma_j = p_i x_i \sum_j \beta_j - p_i \gamma_i \sum_j \beta_j$$

assuming $\sum_j \beta_j = 1$

$$\beta_i y - \beta_i \sum_j p_j \gamma_j = p_i x_i - p_i \gamma_i$$

$$x_i = \gamma_i + \frac{\beta_i y}{p_i} - \frac{\beta_i}{p_i} \sum_j p_j \gamma_j$$

$$x_i = \gamma_i + \frac{\beta_i y}{p_i} - \frac{\beta_i}{p_i} p_i \gamma_i - \frac{\beta_i}{p_i} \sum_{j \neq i} p_j \gamma_j$$

$$\frac{p_i}{\beta_i} x_i = \frac{p_i}{\beta_i} \gamma_i + y - p_i \gamma_i - \sum_{j \neq i} p_j \gamma_j$$

$$\frac{p_i}{\beta_i} x_i - \frac{p_i}{\beta_i} \gamma_i + p_i \gamma_i = y - \sum_{j \neq i} p_j \gamma_j$$

$$v(p) = \sum_i \beta_i \ln \left(\gamma_i + \frac{\beta_i y}{p_i} - \frac{\beta_i}{p_i} \sum_j p_j \gamma_j - \gamma_i \right)$$

$$v(p) = \sum_i \beta_i \left[\ln \left(y - \sum_j p_j \gamma_j \right) + \ln \beta_i - \ln p_i \right]$$

$$\Delta v = \sum_i \beta_i \left[\ln \left(y - \sum_j p'_j \gamma_j \right) + \ln \beta_i - \ln p'_i \right] - \sum_i \beta_i \left[\ln \left(y - \sum_j p_j \gamma_j \right) + \ln \beta_i - \ln p_i \right]$$

$$\Delta v = \sum_i \beta_i \left[\ln \left(y - \sum_j p'_j \gamma_j \right) + \ln \beta_i - \ln p'_i - \ln \left(y - \sum_j p_j \gamma_j \right) - \ln \beta_i + \ln p_i \right]$$

let $c = y - \sum_j p_j \gamma_j$

$$\Delta v = \sum_i \beta_i \left[\Delta \ln c - \Delta \ln p_i \right]$$

$$\Delta v = \Delta \ln c \sum_i \beta_i - \sum_i \beta_i \Delta \ln p_i$$

$$\Delta v = \Delta \ln c - \sum_i \beta_i \Delta \ln p_i$$

$$CS(p) = \sum_i \int_{p_i}^{z_i} \left[\gamma_i + \frac{\beta_i y}{t_i} - \frac{\beta_i}{t_i} t_i \gamma_i - \frac{\beta_i}{t_i} \sum_{j \neq i} p_j \gamma_j \right] dt_i$$

$$CS(p) = \sum_i \int_{p_i}^{z_i} \left[(1-\beta_i)\gamma_i + \frac{\beta_i}{t_i} \left(y - \sum_{j \neq i} p_j \gamma_j \right) \right] dt_i$$

$$CS(p) = \sum_i \left[(1-\beta_i)\gamma_i z_i + \beta_i \left(y - \sum_{j \neq i} p_j \gamma_j \right) \ln z_i - (1-\beta_i)\gamma_i p_i - \beta_i \left(y - \sum_{j \neq i} p_j \gamma_j \right) \ln p_i \right]$$

$$CS(p) = \sum_i \left[(1-\beta_i)\gamma_i z_i + \beta_i \left(\frac{p_i}{\beta_i} x_i - \frac{p_i}{\beta_i}\gamma_i + p_i \gamma_i \right) \ln z_i - (1-\beta_i)\gamma_i p_i - \beta_i \left(y - \sum_{j \neq i} p_j \gamma_j \right) \ln p_i \right]$$

$$CS(p) = \sum_i \left[(1-\beta_i)\gamma_i z_i + (p_i x_i - (1-\beta_i) p_i \gamma_i) \ln z_i - (1-\beta_i)\gamma_i p_i - (p_i x_i - (1-\beta_i) p_i \gamma_i) \ln p_i \right]$$

$$\Delta CS = \sum_i \left[(1-\beta_i)\gamma_i z_i + (p_i' x_i' - (1-\beta_i) p_i' \gamma_i) \ln z_i - (1-\beta_i)\gamma_i p_i' - (p_i' x_i' - (1-\beta_i) p_i' \gamma_i) \ln p_i' \right]$$

$$- \sum_i \left[(1-\beta_i)\gamma_i z_i + (p_i x_i - (1-\beta_i) p_i \gamma_i) \ln z_i - (1-\beta_i)\gamma_i p_i - (p_i x_i - (1-\beta_i) p_i \gamma_i) \ln p_i \right]$$

$$\Delta CS = \sum_i (1-\beta_i)\gamma_i z_i - (1-\beta_i)\gamma_i z_i$$

$$+ \sum_i (p_i' x_i' - (1-\beta_i) p_i' \gamma_i) \ln z_i - (p_i x_i - (1-\beta_i) p_i \gamma_i) \ln z_i$$

$$- \sum_i (1-\beta_i)\gamma_i p_i' - (1-\beta_i)\gamma_i p_i$$

$$- \sum_i (p_i' x_i' - (1-\beta_i) p_i' \gamma_i) \ln p_i' - (p_i x_i - (1-\beta_i) p_i \gamma_i) \ln p_i$$

$$\Delta CS = \sum_i (p_i' x_i' - p_i x_i - (1-\beta_i)(p_i' \gamma_i - p_i \gamma_i)) \ln z_i$$

$$-\sum_i (1-\beta_i)(\gamma_i p_i' - \gamma_i p_i)$$

$$-\sum_i \left(p_i' x_i' - (1-\beta_i) p_i' \gamma_i\right) \ln p_i' - \left(p_i x_i - (1-\beta_i) p_i \gamma_i\right) \ln p_i$$

$$\Delta CS = \sum_i \left((p_i + \Delta p_i)(x_i + \Delta x_i) - p_i x_i - (1-\beta_i)\gamma_i \Delta p_i\right) \ln z_i$$

$$-\sum_i (1-\beta_i)\gamma_i \Delta p_i$$

$$-\sum_i \left((p_i + \Delta p_i)(x_i + \Delta x_i) - (1-\beta_i)(p_i + \Delta p_i)\gamma_i\right) \ln p_i (1+\Delta) - \left(p_i x_i - (1-\beta_i) p_i \gamma_i\right) \ln p_i$$