Optimal Redistribution with a Shadow Economy

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Abstract

We examine the constrained efficient allocations in the Mirrlees (1971) model with a shadow economy. There are two labor markets: formal and informal. The income from the formal market is observed by the planner, while the income from the informal market is not. There is a distribution of workers that differ with respect to the formal and the informal productivity. We show that when the planner does not observe individual productivities some workers may optimally work in the shadow economy. Moreover, the social welfare of the model with the shadow economy can be higher than the welfare of the model without the informal sector. These results hold even when each agent is more productive formally than in the shadow economy. In order to apply our theory, we propose a novel way of estimating workers' productivities in the two sectors from micro data. Calibrating the model to Colombia, where 58% of workers are employed informally, we find that the optimal shadow economy is half that size. The optimal income tax schedule is very different then the one implied by the Mirrlees (1971) model without the informal sector.

1 Introduction

Informal activity, defined broadly as any endeavor which is not necessarily illegal but evades taxation, accounts for a large fraction of economic activity in both developing and developed economies. According to Jutting et al. [2009] more then half of the jobs in the non-agricultural sector worldwide can be considered informal. The share of informal production in the GDP of high income OECD countries in the years 1999-2007 was estimated as 13.5% (Schneider et al. [2011]). Given this, the informal sector should be considered in the design of fiscal policy. This paper extends the theory of the optimal income taxation by Mirrlees [1971] to economies with an informal labor market.

Suppose that individuals can substitute formal, taxable income with hidden income from informal employment. If the planner knows the individual productivities, lump sum taxation is used and each

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agent supplies only the most productive type of labor, which can be formal or informal. When the planner does not know the individual productivities, the optimal allocation involves distortionary taxation and informal labor might emerge, even if it is less productive than formal labor.

The welfare consequences of the existence of a shadow economy are not obvious. The shadow economy affects workers' incentives by providing an alternative dimension of deviating from the planner's allocation. Intuitively, if the shadow economy facilitates deviations of agents the planner wants to tax, it constrains redistribution more than the Mirrlees [1971] model. On the other hand, the shadow economy may be used to make deviations of some agents less attractive, allowing for more redistribution.

Consider a reform that increases the average tax paid at high levels of income in the economy without the informal sector. Such a policy can make the very productive workers work less, have low income (with a lower average tax rate) and more leisure. This reaction of workers to the change in taxes might even lower revenues. A way to ameliorate such a reaction is to raise the marginal tax rates at the low rather than high levels of income. The high marginal tax rates at the low income levels motivate the productive agents to work hard, but they also discourage the labor supply of the less productive workers. This is the famous efficiency-equity trade-off of the Mirrlees model.

The shadow economy modifies this trade-off in two ways. First, it changes the efficiency cost of discouraging the *formal* work of the *formally* less productive agents. If these workers lose little earning potential by moving to the shadows, the efficiency cost is going to be low. Second, the shadow economy changes the utility that workers which are highly productive *formally* can obtain at the low levels of *formal* income. If they are also very productive in the shadow economy, they can complement their lower formal income with the shadow earnings. The relative strength of these two effects determines whether the shadow economy limits or improves the optimum welfare.

We explain our main findings with a simple model of two types and a Rawlsian planner. The informal sector can be a part of the optimum only if agents that are less productive formally (low type) have a *comparative advantage* in shadow labor over the agents that are more productive formally (high type). Otherwise, sending the low type to the shadows will make the deviation of the high type more attractive. The informal sector will be a part of the optimum if additionally the low type is scarce and relatively productive in the shadow economy. The low population share of the low type implies high tax revenue gains from the distortionary taxation. The low productivity loss of this type from moving to the shadow economy means that not an entire earning potential of the displaced agents is lost. Moreover, we derive the shadow productivity thresholds which characterize the impact of existence of shadow economy on the optimum welfare. Depending on which side of thresholds the economy is located, the planner can achieve higher or lower social welfare than in the equivalent economy without the informal labor market.

In order to make our model applicable in policy making, we solve the model with a continuum of types. We derive the full set of optimality conditions, including the conditions for optimal bunching of types at the kinks of the tax schedule. Perhaps surprisingly, we find that the condition for a given type to optimally work in the shadow economy is almost identical to the analogous condition from the simple model. It is optimal for a given type to work in the shadow economy if the productivity loss from doing so is low, the type has low density in comparison to the fraction of types above and the planner wants to efficiently tax the agents with higher income.

Estimation of the joint distribution of formal and informal productivities from the micro data is difficult, because most people work only in one sector. We estimate a factor, which is a linear combination of workers' and jobs' characteristics that explains most of the variability of shadow and formal productivities. We use the factor to match similar individuals and infer their productivities in both sectors.

We calibrate our model to Colombia, where 58% of workers are employed informally. We find that with the Rawlsian planner the optimal shadow economy is half of that size. In comparison the the actual income tax in Colombia, the optimal tax schedule has lower marginal rates at the bottom and higher elsewhere. Efficient taxation allows the planner to increase the transfer to the poorest by 50% while keeping the same budget balance as the actual tax. What is important, the optimal tax schedule is very different then the one implied by the Mirrlees (1971) model without the informal sector. The naive application of the Mirrlees (1971) income tax would displace an excessive number of workers to the shadow economy.

Related literature Tax avoidance has been studied at least since Allingham and Sandmo [1972]. For us, the most relevant paper from this literature was written by Kopczuk [2001]. He derives the optimal linear income tax in a general environment with two-dimensional heterogeneity in productivity and tax avoidance skills. Moreover, he points at the possibility of tax avoidance in the optimum. We, in contrast to Kopczuk, focus on the optimal nonlinear income tax and provide a sharp characterization of the optimal shadow economy.

Alvarez-Parra and Sánchez [2009] study the optimal unemployment insurance with moral hazard in search effort and informal labor market. Hidden employment limits the ability of the planner to use a decreasing future consumption in order to motivate agents to search hard for a job. After a sufficiently long unemployment spell, it is impossible to provide incentives for search, so agents are allowed to stay in the shadow economy. It is another environment with information frictions, in which the informal employment is utilized in the optimal allocation.

Contribution Our contribution is threefold. We are the first to derive the optimal nonlinear income tax schedule of a country with an informal sector. We provide the methodology for estimating formal and informal productivities from micro data. Finally, we show that the application of our theory should yield large welfare gains in Colombia.

Structure of the paper In the next section we illustrate the main ideas of the paper with a simple model. In Section 3 we solve a model with a large number of types and general social preferences. In Section 4 we introduce our methodology of extracting shadow productivities from the micro data and apply it to Colombia. We derive the optimal Colombian tax schedule in Section 5. In the last section we discuss the implications of our findings in the broader context.

2 Simple model

Imagine an economy inhabited by people that share preferences but differ in productivity. There are two types of individuals, indexed by letters l and h, with strictly positive population shares μ_l and μ_h. They all care about consumption c and labor supply n according to the utility function

$$U(c,n) = c - v(n). \tag{1}$$

We assume that v is increasing, strictly convex and twice differentiable. We also impose that $v'(0) = 0$ and $\lim_{n\to\infty} v'(n) = \infty$. We denote the inverse function of v' by g.

There are two labor markets and, correspondingly, each agent is equipped with two linear production technologies. An agent $i \in \{l, h\}$ produces with productivity ϕ_i in a formal labor market, and with productivity ψ_i in an informal labor market. Type h is more productive in the formal market than type l: $\phi_h > \phi_l$. Moreover, in this section we assume that each type's informal productivity is lower than formal productivity. We relax this assumption when we consider the full model.

$$\forall_i \ \phi_i > \psi_i. \tag{2}$$

Any agent may work formally, informally, or in both markets simultaneously. An agent of type i works n_i hours in total, which is the sum of n_i^f hours at the formal job and n_i^s hours in the shadow economy. The formal and the informal income, denoted by y_i^f and y_i^s respectively, is a product of the relevant productivity and the relevant labor supply. The allocation of resources may involve transfers across types, so one's consumption may be different than the sum of formal and informal income. In order to capture these flows of resources, we introduce a tax T_i, equal to the gap between total income and total consumption

$$T_i \equiv y_i^f + y_i^s - c_i. \tag{3}$$

A negative tax is called a transfer, and we are going to use these terms interchangeably.

The social planner follows John Rawls' theory of justice and wants to improve the well-being of the least well-off agents,[1] but is limited by imperfect knowledge. The planner knows the structure and parameters of the economy, but, as in the standard Mirrlees model, does not observe the type of any individual. In addition, shadow income and labor are unobserved by the planner as well. The only variables at the individual level the planner sees and can directly verify are the formal income y_i^f and the tax T_i. We can think about y_i^f and $y_i^f - T_i$ as a pre-tax and an after-tax reported income. Although shadow labor cannot be controlled directly, it is influenced by the choice of formal labor. Formal labor affects the marginal disutility from labor and hence changes the agent's optimal choice of shadow hours. Two types of labor are related according to the following function, implied by the agent's first order condition

$$n_i^s\left(n^f\right) = \max\left\{g(\psi_i) - n^f, 0\right\}.$$

[1] We pick this particular point of the Pareto frontier because it allows us to show the interesting features of the model with relatively easy derivations. At the end of this section we discuss how other constrained efficient allocations look like.

When the agent works a sufficient number of hours in the formal sector, the marginal disutility from labor is too high to work additionally in the shadows. However, if the formal hours fall short of $g(\psi_i)$, the resulting gap is filled with shadow labor.

The planner maximizes the Rawlsian social welfare function, given by a utility level of the worst-off agent

$$\max_{\left\{\left(n_i^f, T_i\right) \in \mathbb{R}_+ \times \mathbb{R}\right\}_{i \in \{l,h\}}} \min \left\{U\left(c_l, n_l\right), U\left(c_h, n_h\right)\right\}, \tag{4}$$

subject to the relation between formal and shadow labor

$$n_i^s\left(n^f\right) = \max\left\{g\left(\psi_i\right) - n^f, 0\right\}, \tag{5}$$

the accounting equations

$$\forall_{i \in \{l,h\}}\; c_i = \phi_i n_i^f + \psi_i n_i^s\left(n_i^f\right) - T_i, \tag{6}$$

$$\forall_{i \in \{l,h\}}\; n_i = n_i^f + n_i^s\left(n_i^f\right), \tag{7}$$

a resource constraint

$$\sum_{i \in \{l,h\}} \mu_i T_i \geq 0, \tag{8}$$

and incentive-compatibility constraints

$$\forall_{i \in \{l,h\}}\; U\left(c_i, n_i\right) \geq U\left(\phi_{-i} n_{-i}^f + \psi_i n_i^s\left(\frac{\phi_{-i}}{\phi_i} n_{-i}^f\right) - T_{-i}, \frac{\phi_{-i}}{\phi_i} n_{-i}^f + n_i^s\left(\frac{\phi_{-i}}{\phi_i} n_{-i}^f\right)\right). \tag{9}$$

We denote the generic incentive constraint by $IC_{i,-i}$. It means that an agent i cannot be better off by earning the formal income of the other type and simultaneously adjusting informal labor.

2.1 First-best

What if the planner is omniscient and directly observes all variables? The planner knows types and can choose the shadow labor supply directly. The optimal allocation is a solution to the welfare maximization problem (4) where planner chooses both formal and shadow labor and a tax of each type subject only to the accounting equations (6) and (7) and the resource constraint (8). All types are more productive in the formal sector than in the shadow economy, so no agent will work informally. Each agent will supply the formal labor efficiently, equalizing the marginal social cost and benefit of working. Moreover, the planner redistributes income from h to l in order to achieve the equality of well-being.

Proposition 1. *In the first-best both types work only formally and supply an efficient amount of labor: $\forall_i v'\left(n_i\right) = \phi_i$. Utility levels of the two types are equal: $U\left(c_l, n_l\right) = U\left(c_h, n_h\right)$.*

We can slightly restrict the amount of information available to the planner without affecting the optimal allocation. Suppose that the planner still observes the formal productivity, but shadow labor and income are hidden. The optimal allocation is a solution to (4) subject to the relation between shadow and formal labor (5), the accounting equations (6) and (7) and the resource constraint (8).

Proposition 2. *If the planner knows types, but does not observe shadow labor and income, the planner can achieve the first-best.*

When the types are known, the planner can use the lump-sum taxation and implement the first-best. Without additional frictions, the hidden shadow economy does not constrain the social planner.

2.2 Second-best

Let's consider the problem in which neither type nor informal activity is observed. The planner solves (4) subject to all the constraints (5) - (9). We call the solution to this problem the second-best or simply the optimum.

In the first-best, both types work only on the formal market and their utilities are equal. If h could mimic the other type, higher formal productivity would allow h to increase utility. Hence, the first-best does not satisfy $IC_{h,l}$ and this constraint limits the welfare at the optimum. On the other hand, $IC_{l,h}$ never binds at the optimum. It would require the redistribution of resources from type l to h, which is clearly suboptimal.

Proposition 3. *The optimum exists and is not the first-best.* $IC_{h,l}$ *is binding, while* $IC_{l,h}$ *is slack.*

2.2.1 Optimal shadow economy

The standard Mirrlees model typically involves labor distortions, since they can relax the binding incentive constraints. If type i is tempted to pretend to be of the type $-i$, distorting number of hours of $-i$ will discourage the deviation. Agents differ in labor productivity, so if i is more (less) productive than the other type, decreasing (increasing) number of hours worked by $-i$ will make the deviation less attractive. Proposition 3 tells us that no agent wants to mimic type h, hence the planner has no reason to distort the labor choice of these agents. Moreover, according to (5) shadow labor is supplied only if formal labor is sufficiently distorted. Hence, the classic result of no distortions at the top implies here that h will work only formally.

Corollary 1. *Type h faces no distortions and never works in the shadow economy.*

On the other hand, the planner can improve social welfare by distorting the formal labor supply of type l. Stronger distortions relax the binding incentive constraint and allow the planner to redistribute more. If distortions are strong enough, type l will end up supplying shadow labor. Optimality of doing so depends on whether and by how much increasing shadow labor of type l relaxes the binding incentive constraint. As Proposition 4 demonstrates, a comparative advantage of type l in shadow labor plays a crucial role. In the proof we use the optimality condition derived in the Appendix (see Lemma 5 in the Appendix 2). In order to make sure that this condition is well behaved, we require that v'' is nondecreasing.[2]

[2] In the canonical case of isoelastic utility, it means that the elasticity of the labor supply is not greater than 1.

Proposition 4. *Suppose that v'' is nondecreasing. Type l may optimally work in the shadow economy only if*

$$\left(\frac{\psi_l}{\phi_l} - \frac{\psi_h}{\phi_h}\right)\mu_h \geq \frac{\phi_l - \psi_l}{\phi_l}\mu_l. \tag{10}$$

Condition (10) is also a sufficient condition for type l to optimally work in the shadow economy if $\frac{\phi_h}{\phi_l} g(\psi_h) \geq g(\psi_l)$. Otherwise, the sufficient (but not necessary) condition is

$$\left(\frac{\psi_l}{\phi_l} - \frac{v'\left(\frac{\phi_l}{\phi_h} g(\psi_l)\right)}{\phi_h}\right)\mu_h \geq \frac{\phi_l - \psi_l}{\phi_l}\mu_l. \tag{11}$$

Inequality (10) provides a necessary condition for the optimal shadow economy by comparing the marginal benefit and cost of increasing shadow labor of type l. The left hand side is the comparative advantage of type l over type h in the shadow labor, multiplied by the share of type h. This advantage has to be positive for type l to optimally work in the shadow economy. Otherwise, increasing shadow labor of this type does not relax the binding incentive constraint. Since the shadow economy does not facilitate screening of types, there are no benefits from the productivity-inferior shadow sector. The welfare gains from the relaxed incentive constraint are proportional to the share of type h, as the planner obtains more resources for redistribution by imposing a higher tax on this type. On the right hand side, the cost of increasing shadow labor is given by the productivity loss from using the inferior shadow production, multiplied by the share of types that supply shadow labor.

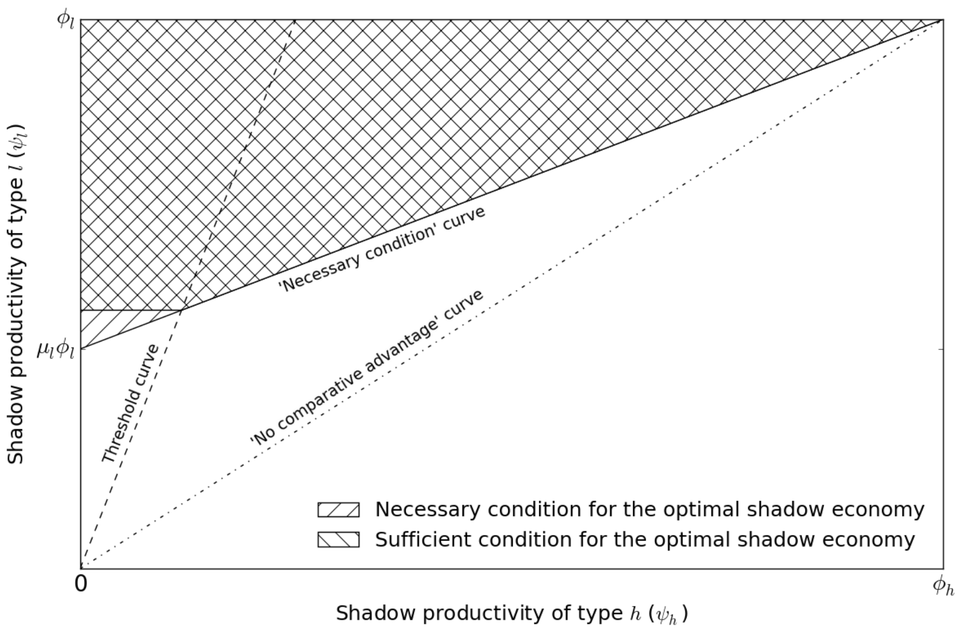
Condition (10) is also a sufficient condition for type l to work in the shadow economy if the shadow productivity of type h is not much lower than the shadow productivity of type l. If that is not the case, the optimality condition derived in Lemma 5 (see Appendix 2) is not sufficient and we have to impose a stronger sufficiency condition (11).

Figure 1 illustrates the proposition on the diagram of the parameter space (ψ_h, ψ_l). Along the the diagonal no type has the comparative advantage, since ratios of shadow and formal productivity of the two types are equal. The optimal shadow economy requires that type l has the comparative advantage in shadow labor, so all the interesting action happens above the diagonal. Inequality (10) determines 'Necessary condition' curve. All points above this line satisfy (10) and only in this region type l may optimally supply shadow labor. Condition (10) is both necessary and sufficient for the optimal shadow economy to the right of Threshold curve. To the left of it, the sufficiency region is determined by inequality (11). Note that 'Necessary condition' curve crosses the vertical axis at the value $\mu_l\phi_l$. As the proportion of type l decreases toward zero, the region where shadow economy is optimal increases, in the limit encompassing all the points where type l has the comparative advantage over h in shadow labor.

Now we know when type l optimally works in the shadow economy. Proposition 5 tells us, how much shadow labor should type l supply in this case.

Proposition 5. *Suppose that type l optimally works in the shadow economy. Type l works only in the shadow economy if $\psi_h \geq \psi_l$. Type l works in both sectors simultaneously if $\psi_h < \psi_l$.*

Figure 1: The optimal shadow economy



When type l is more productive in the shadows than h and works only in the shadow economy, then by $IC_{h,l}$ the utility of type l will be greater than the utility of h. Since the planner is Rawlsian, the utility levels of both types will be equalized by making type l work partly in the formal economy. On the other hand, when type h is more productive informally, $IC_{h,l}$ means that the utility of type l will be always lower. Then if the shadow economy benefits type l, the planner will use it as much as possible.

2.2.2 Shadow economy and welfare

In order to examine the welfare implications of the shadow economy, we compare social welfare of the two allocations: the optimum of the standard Mirrlees model (noted with a superscript ^{M}) and the optimum of the shadow economy model (noted with a superscript ^{SE}). We can think about the standard Mirrlees model as a special case of our model, in which both ψ_l and ψ_h are equal 0. We are going to use the utility of type l as a measure of welfare. Hence, we implicitly assume that in the optimum the planner does not equalize utilities of two types. If the planner is able and willing to do so, the existence of the shadow economy is clearly welfare improving in comparison to the Mirrlees model where type h has always greater utility than type l.

Let's consider the case in which the existence of shadow economy improves welfare. It can happen only if the shadow economy is used in the optimum - otherwise the planner could implement the same allocation in the standard Mirrlees model. Moreover, if in the optimum type l works in the shadow economy, this type will supply only shadow labor.[3] We can decompose the difference in

[3]Recall that we ruled out cases in which the planner equalizes the utility of both types in the optimum (by Proposition 5 it happens when the shadow economy is optimal and $\psi_l \geq \psi_h$). Only in such cases it may be necessary

welfare between the two allocations in the following way

$$\underbrace{U\left(c_l^{SE}, n_l^{SE}\right) - U\left(c_l^M, n_l^M\right)}_{\text{total welfare gain}} = \underbrace{U\left(\psi_l n_l^{SE}, n_l^{SE}\right) - U\left(\phi_l n_l^M, n_l^M\right)}_{\text{efficiency gain}} + \underbrace{T_l^M - T_l^{SE}}_{\text{redistribution gain}}. \tag{12}$$

The efficiency gain measures the difference in distortions imposed on type l, while the redistribution gain describes the change in the level of transfer type l receives. Thanks to the quasilinear preferences, we can decompose these two effects additively.

Efficiency gain The distortion imposed on type l in the shadow economy arise from the productivity loss $\phi_l - \psi_l$. By varying ψ_l, this distortion can be made arbitrarily small. On the other hand, the distortion of the standard Mirrlees model is implied by the marginal tax rate on formal income. Given redistributive social preferences, it is always optimal to impose a positive tax rate on type l. The efficiency gain, which captures the difference in distortions between two regimes, is strictly increasing in ψ_l.

Redistribution gain The shadow economy improves redistribution if the planner is able to give higher transfer to type l (or equivalently raise higher tax from type h). The difference in transfers can be expressed as

$$T_l^M - T_l^{SE} = \mu_h \left(U\left(\phi_l n_l^M, \frac{\phi_l}{\phi_h} n_l^M\right) - U\left(\psi_h n_h^{SE}, n_h^{SE}\right)\right).$$

What determines the magnitude of redistribution is the possibility of production of type h after misreporting. In the standard Mirrlees model deviating type h uses formal productivity and can produce only as much output as type l. In the allocation where type l works only informally, type h cannot supply any formal labor, but is unconstrained in supplying informal labor. Hence, the redistribution gain is strictly decreasing in ψ_h.

Proposition 6 uses the decomposition into the efficiency and redistribution gains in order to derive threshold values for shadow productivity of each type. Depending on which side of the thresholds the productivities are, the existence of the shadow economy improves or deteriorates social welfare in comparison to the standard Mirrlees model.

Proposition 6. *Define an increasing function $H(\psi) = U(\psi g(\psi), g(\psi))$ and the following threshold values*

$$\bar{\psi}_l = H^{-1}\left(U\left(\phi_l n_l^M, n_l^M\right)\right) \in (0, \phi_l), \qquad \bar{\psi}_h = H^{-1}\left(U\left(\phi_l n_l^M, \tfrac{\phi_l}{\phi_h} n_l^M\right)\right) \in (0, \phi_h). \tag{13}$$

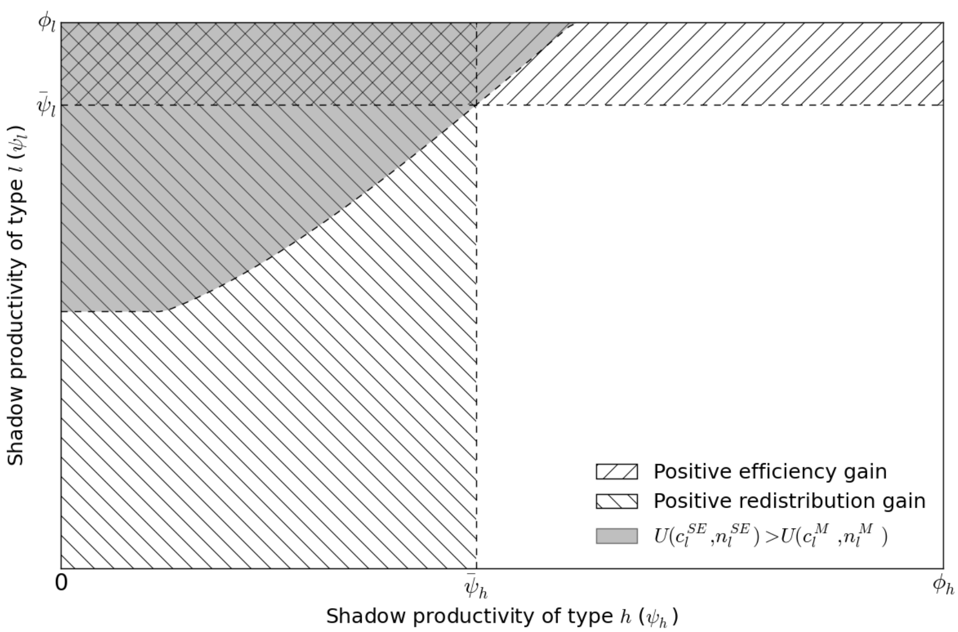
If $\psi_l \geq \bar{\psi}_l$ and $\psi_h \leq \bar{\psi}_h$, where at least one of these inequalities is strict, the existence of the shadow economy improves welfare in comparison to the standard Mirrlees model.

for optimality that type l optimally supplies both formal and shadow labor.

If $\psi_l \leq \bar{\psi}_l$ and $\psi_h \geq \bar{\psi}_h$, where at least one of these inequalities is strict, the existence of the shadow economy deteriorates welfare in comparison to the standard Mirrlees model.

The proposition is illustrated on the Figure 2. When the shadow productivity of type l is above $\bar{\psi}_l$, the efficiency gain is positive. The redistribution gain is positive when the shadow productivity of type h is below $\bar{\psi}_h$. Obviously, when both gains are positive, the shadow economy benefits welfare. On the other hand, when both gains are negative, the shadow economy is a burden and decreases welfare. Intuitively, the shadow economy does not have to strengthen both redistribution and efficiency simultaneously to be welfare improving. Particularly interesting is the region where the redistribution gain is negative, but the efficiency gain is sufficiently high such that the welfare is higher with the shadow economy. In this case the optimum of the shadow economy model Pareto dominates the optimum of the Mirrlees model. Type l gains, since the welfare is higher with the shadow economy. Type h benefits as well, as the negative redistribution gain implies a lower tax of this type.

Figure 2: Shadow economy and welfare



2.2.3 General social preferences

In this short section we will derive some properties of the whole Pareto frontier of the two-types model. We consider the planner that maximizes the general utilitarian social welfare function

$$\lambda_l \mu_l U(c_l, n_l) + \lambda_h \mu_h U(c_h, n_h),$$

where the two Pareto weights are non-negative and sum up to 1. The maximization is subject to the constraints (5) - (9).

From the Rawlsian case we know that the comparative advantage of type l in shadow labor is necessary for this type to work in the shadows. Proposition 7 generalizes this observation.

Proposition 7. *Type $i \in \{l, h\}$ may optimally work in the shadow economy only if $\frac{\psi_i}{\phi_i} > \frac{\psi_{-i}}{\phi_{-i}}$ and $\lambda_i > \lambda_{-i}$.*

In order to optimally work in the shadow economy, any type $i \in \{l, h\}$ has to satisfy two requirements. First, type i needs to have the comparative advantage in the shadow labor over the other type. Otherwise, shifting labor from formal to shadow sector does not relax the incentive constraints. Second, the planner has to be willing to redistribute resources to type i - the Pareto weight of this type has to be greater than the weight of the other type. The shadow economy can be beneficial only when it relaxes the binding incentive constraints, and the incentive constraint $IC_{-i,i}$ binds if $\lambda_i > \lambda_{-i}$. Intuitively, if the planner prefers to tax rather than support some agents, it is suboptimal to let them evade taxation.

When will type i optimally work in the shadow economy? Let's compare the welfare of two allocations. In the first allocation (denoted by superscript ^{SE}) type i works exclusively in the shadow economy. It provides the lower bound on welfare when type i is employed informally. The second allocation (denoted by ^{M}) is the optimum of the standard Mirrlees model, or equivalently the optimum of the shadow economy model where $\psi_i = \psi_{-i} = 0$. It is the upper bound on welfare when type i is employed only in the formal sector. We can decompose the welfare difference between these two allocations in the familiar way

$$\underbrace{W^{SE} - W^{M}}_{\text{total welfare gain}} = \underbrace{\mu_i \lambda_i \left(U\left(\psi_i n_i^{SE}, n_i^{SE}\right) - U\left(\phi_i n_i^{M}, n_i^{M}\right)\right)}_{\text{efficiency gain}} + \underbrace{\mu_i \left(\lambda_i - \lambda_{-i}\right)\left(T_i^{M} - T_i^{SE}\right)}_{\text{redistribution gain}}. \tag{14}$$

The welfare difference can be decomposed into the difference in effective distortions imposed on type i and the difference in transfers received by this type. The only essential change in comparison to the simpler Rawlsian case given by (12) comes from the Pareto weights. The more the planner cares about type $-i$, the less valuable are gains in redistribution in comparison to the gains in efficiency.

Proposition 8. *Suppose that $\lambda_i > \lambda_{-i}$ for some $i \in \{l, h\}$. Define the following thresholds*

$$\bar{\psi}_i = H^{-1}\left(U\left(\phi_i n_i^{M}, n_i^{M}\right)\right) \in (0, \phi_i), \qquad \bar{\psi}_{-i} = H^{-1}\left(U\left(\phi_i n_i^{M}, \frac{\phi_i}{\phi_{-i}} n_i^{M}\right)\right) \in (0, \phi_{-i}).$$

If $\psi_i \geq \bar{\psi}_i$ and $\psi_{-i} \leq \bar{\psi}_{-i}$, where at least one of these inequalities is strict, then type i optimally works in the shadow economy and the optimum welfare is strictly higher than in the standard Mirrlees model.

Proposition 8 generalizes the thresholds from Proposition 6. Interestingly, when the planner cares more about the more productive formally type h, these agents may end up working in the shadow economy. It may be surprising, since in the standard Mirrlees model the formal labor supply of this type is optimally either undistorted, or distorted upwards, while supplying shadow labor requires a downwards distortion. Nevertheless, if shadow economy magnifies productivity differences between types, it may be in the best interest of type h to supply only informal labor and enjoy higher transfer financed by the other type. The shadow economy in such allocation works as a tax haven, accessible only to the privileged.

3 Full model

In this section we describe how to find an optimal tax schedule in an economy with a large number of types. We are going to focus on preferences without wealth effects. Agents' types are distributed on the interval $[0,1]$ according to a density μ_i and a cumulative density M_i. The density μ_i is atomless. We assume that formal and informal productivities are functions which are differentiable with respect to type and denote these derivatives by $\dot{\phi}_i$ and $\dot{\psi}_i$. We sort types such that formal productivity is increasing in i.

Agents' preferences are described by the utility function $U(c,n) = c - v(n)$, where v is increasing, strictly convex and twice differentiable. Let $V_i\left(y^f, T\right)$ be the indirect utility function of an agent of type i whose reported income is y^f and who pays a tax T:

$$V_i\left(y^f, T\right) \equiv \max_{n^s \geq 0} y^f + \psi_i n^s - T - v\left(\frac{y^f}{\phi_i} + n^s\right). \tag{15}$$

In addition to earning the formal income, the agent is optimally choosing the amount of informal labor. Due to concavity of the problem, we can express V more explicitly as

$$V_i\left(y^f, T\right) = y^f + \psi_i n_i^s - T - v\left(\frac{y^f}{\phi_i} + n_i^s\right), \ \min\left\{v'\left(n_i\right) - \psi_i, n_i^s\right\} = 0. \tag{16}$$

Whenever the tax distortions imposed on some agent are small, no informal labor is supplied. When the distortions are sufficiently severe, or in other words the marginal tax rate t_i is sufficiently high, agent seeks informal employment.

The planner maximizes a weighted, concave transformation of agents' utilities

$$\max_{\left(y_i^f, T_i\right)_{i=0}^1} \int_0^1 \lambda_i G\left(V_i\left(y_i^f, T_i\right)\right) d\mu_i, \tag{17}$$

where G is increasing, concave and differentiable, while the welfare weights $\left(\lambda_i\right)_{i=0}^1$ integrate to 1. The distribution of welfare weights is atomless.[4] The planner is constrained by available resources, where E denotes some fixed expenditures

$$\int_0^1 T_i d\mu_i \geq E. \tag{18}$$

[4]It's easy to relax this assumption and we are going to do so in the quantitative exercise, where we consider, among other social welfare functions, the Rawlsian planner.

Moreover, the tax schedule has to satisfy incentive compatibility

$$\forall_{i,i'\in[0,1]} V_i\left(y_i^f, T_i\right) \geq V_i\left(y_{i'}^f, T_{i'}\right), \tag{19}$$

which means that no agent can gain by mimicking any other type. The allocation which solves (17) subject to (18) and (19) is called the second-best or the optimum.

3.1 Incentive-compatibility

In the standard Mirrlees model, the single crossing property of agents' preferences allows the planner to focus only on local incentive compatibility constraints. The single-crossing means that if we fix the marginal tax rate, a higher type is willing to work more than a lower type. This property in the context of our model implies that the formal income selected by an agent is a nondecreasing function of type.

Lemma 1. *The indirect utility function V satisfies the single crossing condition if $\frac{d}{di}\left(\frac{\psi_i}{\phi_i}\right) < 0$.*

The single-crossing holds when the agents with lower formal productivity have a comparative advantage in working in the informal sector. From now onwards we assume that this is the case: $\frac{\psi_i}{\phi_i}$ is decreasing with type. The single-crossing allows us to replace the complicated incentive compatibility condition (19) with two simpler requirements.

Proposition 9. *The allocation $\left(y_i^f, T_i\right)_{i\in[0,1]}$ is incentive-compatible if and only if y_i^f is nondecreasing in type and $\frac{d}{dj}V_i\left(y_j^f, T_j\right)\Big|_{j=i} = 0$ whenever $\frac{d}{di}y_i^f$ exists. The utility schedule $V_i \equiv V_i\left(y_i^f, T_i\right)$ of the incentive compatible allocation is continuous everywhere, differentiable almost everywhere and for any $i < 1$ can be expressed as*

$$V_i\left(y_i^f, T_i\right) = V_0\left(y_0^f, T_0\right) + \int_0^i \dot{V}_j dj, \tag{20}$$

where

$$\dot{V}_i \equiv \left(\frac{\dot{\phi}_i}{\phi_i} n_i^f + \frac{\dot{\psi}_i}{\psi_i} n_i^s\right) v'\left(n_i\right). \tag{21}$$

The single crossing implies that higher types choose higher formal income. Hence, assigning a lower income to a higher type would violate incentive compatibility. It is enough to focus just on local deviations: no agent should be able to improve utility by marginally changing the formal income. Note that the formal income may be, and often will be, discontinuous in type. Nevertheless, the indirect utility function preserves some smoothness and can be expressed as an integral of its marginal increments.

Let's call $\dot{V}_i$ the *marginal information rent* of type i. It shows how the utility level changes with type. The higher the average rate of productivity growth, weighted by the labor inputs in two sectors, the faster utility increases with type. We will use perturbations in $\dot{V}$ to derive optimality conditions.

3.2 Optimality conditions

Doligalski [2015] derives the full set of optimality conditions in the standard Mirrlees model, including the optimal income tax kinks.[5] We extend this approach to the model with the shadow economy. The shadow economy makes it more likely that the optimal allocation will involve bunching of different types at the kinks of the income tax schedule.

We obtain the interior optimality conditions by assuring that the social welfare cannot be improved by perturbing the marginal information rent of any type.[6] Decreasing the slope of the utility schedule at type i results in an increase of tax distortions of this type, which is costly. On the other hand, by (20) the perturbation shift downwards the entire utility schedule above type i, which is equivalent to increasing a non-distortionary tax. The optimality condition balances the cost of distortions with the benefit of efficient taxation. The shadow economy enters the picture by affecting the cost of increasing distortions of agents that work informally. If the income schedule implied by the interior optimality conditions is nondecreasing then, under mild sufficiency conditions listed in Theorem 1, the interior allocation is optimal.

If the interior optimality conditions result in the locally decreasing income schedule, the interior conditions fail to be necessary for the optimum, since they lead to the allocation that is not incentive-compatible. The optimal incentive-compatible allocation will involve bunching different types at the kinks of the tax schedule. The optimal kinks balance the cost of distortions of bunched agents with the benefit of non-distortionary taxation of all agents above the kink. The shadow economy makes the interior optimality conditions more likely to load to a decreasing formal income schedule. Whenever it is optimal for a given agent to supply some informal labor, interior conditions imply that the formal income of this agent should be reduced to zero. Since this can happen for agents in the middle of the formal income distribution, the shadow economy may produce kinks in the optimal income tax.

Interior optimality conditions The benefit of shifting the utility schedule of type j without affecting its slope is given by the standard expression

$$N_j \equiv (1 - w_j)\,\mu_j, \text{ where } w_j = \frac{\lambda_j}{\eta} G'\left(V_j\left(y_j^f, T_j\right)\right). \tag{22}$$

A marginal increase of non-distortionary taxation leads to one-to-one increase of tax revenue. On the other hand, it reduces the social welfare, since the utility of type j falls. This impact is captured by the marginal welfare weight w_j. The marginal welfare weight w_j captures the marginal welfare impact of providing type j with one additional unit of consumption, where η is the Lagrange multiplier of the resource constraint.

We assumed that there are no wealth effects, so the non-distortionary tax does not affect the labor choice of agents. Consequently, the term N_j does not depend on whether type j is working informally.

[5] The latest version of the paper can be found here: https://dl.dropboxusercontent.com/u/19338650/papers/optimal_kinks.pdf
[6] Brendon [2013] was the first to use this approach in the Mirrlees model.

The cost of distorting some agent's allocation depends on the involvement of this agent in the shadow activity. Types can be grouped into three sets:

$$\begin{array}{ll} \text{formal workers} & \mathcal{F} = \left\{ i \in [0,1] : v'\left(n_i^f\right) > \psi_i \right\}, \\ \text{marginal workers} & \mathcal{M} = \left\{ i \in [0,1] : v'\left(n_i^f\right) = \psi_i \right\}, \\ \text{shadow workers} & \mathcal{S} = \left\{ i \in [0,1] : v'\left(n_i^f\right) < \psi_i \right\}. \end{array}$$

The formal workers supply only formal labor and their net marginal wage $(1 - t_i)\,\phi_i$ is strictly higher than what they could get in the shadow economy. The marginal workers also supply only the formal labor, but their net marginal wage is equal to their informal productivity. Finally, the shadow workers are employed informally, although they can also supply some formal labor.

The formal workers act exactly like agents in the standard Mirrlees model. The cost of increasing their distortions is given by

$$D_i^f \equiv \frac{t_i}{1 - t_i} \left(\frac{\dot{\phi}_i}{\phi_i} \left(1 + \frac{1}{\zeta_i} \right) \right)^{-1} \mu_i. \tag{23}$$

The cost of increasing distortions depends positively on the marginal tax rate of this type. The marginal tax rate tell us how strongly a reduction of the formal income influences the tax revenue. Moreover, the cost increases with the elasticity of labor supply ζ_i and is proportional to the density of the distorted type.

The perturbation of the marginal information rent works differently for the shadow workers. They supply shadow labor in the quantity that satisfies $v'\left(n_i^f + n_i^s\right) = \psi_i$, which means that their total labor supply n_i is constant. By distorting the formal income, the planner simply shift labor of these types from the formal to the informal sector. As a result, the cost of increasing distortions does not depend on the elasticity of labor supply, but rather on the sectoral productivity differences,

$$D_i^s \equiv \frac{\phi_i - \psi_i}{\psi_i} \left(\frac{\dot{\phi}_i}{\phi_i} - \frac{\dot{\psi}_i}{\psi_i} \right)^{-1} \mu_i. \tag{24}$$

The first term is the relative productivity difference between formal and informal sector. The second term describes how effectively the planner can manipulate the agent's marginal information rent by discouraging the formal labor supply. By the single-crossing assumption, this term is always positive. The density μ_i aggregates the expression to include all agents of type i. Note that this term depends only on the fundamentals of the economy.

The marginal workers are walking a tightrope between their formal and shadow colleagues. If the planner marginally reduces their income, they become the shadow workers. If the planner lifts distortions, they join the formal workers. The cost of changing distortions of these types depends on the direction of perturbation and is equal to either D_i^f or D_i^s.

Having all the cost and benefit terms ready, we can derive the interior optimality conditions. In the optimum, the planner cannot increase the social welfare by varying the marginal information rents of any type. For the formal workers, this means that

$$\forall_{i \in \mathcal{F}} \; D_i^f = \int_i^1 N_j dj. \tag{25}$$

It is a standard optimality condition from the Mirrlees model. The shadow economy does not affect the tax rate of formal agents directly. It may affect them only indirectly, by changing the allocation and the marginal welfare weights of types above.

For the marginal workers it must be the case that increasing tax distortions is beneficial as long as they work only formally, but it is too costly when they start to supply the shadow labor.

$$\forall_{i \in \mathcal{M}} \; D_i^s > \int_i^1 N_j dj \geq D_i^f. \tag{26}$$

The marginal workers do not supply informal labor, but in their case the shadow economy constitutes a binding constraint for the planner. Absent the shadow economy, the planner would set their marginal tax rates at a higher level. In our model, the planner cannot do it, because it would send the marginal workers to informal employment, which is too costly.

Recall that the cost of distorting the shadow worker is fixed. Moreover, the benefit of distorting one particular worker, given by (22), is fixed as well, since the perturbation of the marginal information rent of i has an infinitesimal effect on the allocation of types above. If the planner finds it optimal to decrease the formal income of agent i so much that i starts supplying informal labor, it will be optimal to decrease the formal income even further, until i works only in the shadow economy:

$$\forall_{i \in \mathcal{S}} \; \int_i^1 N_j dj \geq D_i^s, \; y_i^f = 0. \tag{27}$$

The conditions above determine the slope of the utility schedule at each type. What is left is finding the optimal level. Suppose that the planner vary the tax paid by the lowest type, while keeping all the marginal rates fixed. Optimum requires that such perturbation cannot improve welfare:

$$\int_0^1 N_j dj = 0. \tag{28}$$

The conditions we derived above are meaningful only if they do not violate the incentive-compatibility, i.e. lead to income schedule which is never decreasing. They become sufficient, if they pin down a unique allocation. This happens when the cost of distortions is increasing in the amount of distortions imposed, which is implied by the regularity conditions listed in the Theorem 1.

Theorem 1. *If the conditions (25)-(28) always imply the formal income schedule that is nondecreasing in type, they are necessary for the optimum. If in addition the elasticity of labor supply is non-increasing in labor and* $\forall_i \frac{\dot{\phi}_i}{\phi_i} + \frac{\dot{\psi}_i}{\psi_i}\zeta_i \geq 0$*, they are sufficient for the optimum.*

When is the implied formal income schedule nondecreasing? We cannot say much about the income schedule of the formal workers. If the distortion cost (23) falls rapidly relative to the benefit, the interior optimality condition can imply a fall in formal income. But we have the sharp characterization of other types of workers. The marginal workers face the tax rate which equates the return to labor in the two sectors. If the shadow productivity does not fall rapidly in type, the formal income of these agents will be weakly increasing. Finally, by (27) the shadow workers should not work formally at all. Hence, if there is some shadow worker of type higher than any formal or marginal worker, the income schedule will be decreasing.

Lemma 2. *The formal income schedule y^f implied by conditions (25)-(28) is nondecreasing on set M if and only if $\forall_i \frac{\dot{\phi}_i}{\phi_i} + \frac{\dot{\psi}_i}{\psi_i}\zeta_i \geq 0$. The schedule y^f is nondecreasing on the closure of set $\mathcal{S}$ if and only if $\mathcal{S}$ is convex (possibly empty) and located at the bottom of the type space.*

Optimal kinks Whenever conditions (25)-(28) describe the formal income schedule that is decreasing for some types, they fail to be necessary for the optimum. Under the single-crossing assumption we have made, the decreasing formal income is a violation of incentive compatibility. The interior conditions fail to be necessary not only for the types for which the formal income is decreasing, but also for types *above* and *below* them.

Suppose that (25)-(28) imply an income schedule $\bar{y}^f$, which is decreasing on the set of types $\left[\bar{a}, \bar{b}\right]$ and nondecreasing elsewhere. Since the allocation of types $\left(\bar{a}, \bar{b}\right]$ violates incentive compatibility, it has to be changed. We can simply lift the formal income schedule such that it becomes flat at $\left[\bar{a}, \bar{b}\right]$. Note that then we have to lift the formal income of some types above $\bar{b}$, in order to make sure that the schedule is not decreasing there. Let's say that $\bar{c}$ is the first type above $\bar{b}$ for which $\bar{y}^f_{\bar{c}} = \bar{y}^f_{\bar{a}}$. Then this correction of the income schedule involves all types $(\bar{a}, \bar{c})$. After the correction, the allocation of these types clearly does not satisfy conditions (25)-(28). Let's call the "corrected" formal income schedule $\tilde{y}^f$. Note that the types $[\bar{a}, \bar{c}]$ are bunch together at the kink of the income schedule corresponding to $\tilde{y}^f$. In other words, the marginal tax rate increases discontinuously at the income level $\tilde{y}^f_{\bar{a}}$.

The new schedule $\tilde{y}^f$ is incentive compatible. However, in general it is not optimal. The formal income of types $[0, \bar{a}]$ is set according to conditions (25)-(28). These conditions do not take into an account the restriction of the formal income being nondecreasing which is binding for types above. Marginally decreasing $\bar{y}^f_{\bar{a}}$ allows the planner to decrease the formal income of all the constrained types $(\bar{a}, \bar{c})$, which is what the planner wanted to do in in the first place. An infinitesimal cost of distorting $\bar{a}$ results in a much larger welfare gain coming from closing the gap between cost and benefit for the whole segment $(\bar{a}, \bar{c})$. It suggests that the planner will be willing to sacrifice the local optimality conditions of some types below (and including) $\bar{a}$ as well.

Below we apply the optimality condition with respect to the distortions at the kink, derived by Doligalski [2015]. As in the interior case, it balances the cost of distortions of agents at the given income level with the benefits of efficient taxation of agents above. However, there is a whole interval of types that we distort at the same time. Since we change their allocation in the identical way, we will not be able to keep the utility level of all of them constant. As a result, increasing distortions is costly both in terms of fiscal revenue and in terms of welfare of distorted agents.

Suppose that an interval of agents $[a, c]$ is bunched at the kink. Let's marginally decrease the formal income of agents $[a, c)$ and adjust their total tax paid such that the utility of type a is unchanged.

In this way we preserve the continuity of the utility schedule at a. This perturbation will decrease the utility level of all the other types at the kink. We will normalize the perturbation such that we obtain a unit change of the utility of the last agent at the kink. The total cost of this perturbation is given by

$$D_{a,c}^{K} \equiv \left(t_a + \mathbb{E}\left\{\Delta MRS_j w_j \middle| c > j \geq a\right\}\right) \frac{M_c - M_a}{t_c - t_a}, \tag{29}$$

$$\text{where } \Delta MRS_j = \frac{v'(n_a)}{\phi_a} - \frac{v'(n_j)}{\phi_j}.$$

The expression within the brackets is an average impact of a unit perturbation of the formal income. The brackets contain two components: a fiscal and a welfare loss. The fiscal loss from reducing the formal income of each bunched agent is just the marginal tax rate below the kink. The welfare loss is an average marginal welfare weight corrected by a measure of distance of a given agent from type a. The relevant measure of distance in this case is the difference in marginal rates of substitution. The larger ΔMRS_j is, the more type j suffers from the perturbation. By the single crossing condition, ΔMRS_j is increasing in j. Note that ΔMRS_c is just equal $t_c - t_a$. Hence, in order to normalize the impact of perturbation on the types above the kink, we divide the brackets by $t_c - t_a$ We aggregate this average effect by multiplying it by the fraction of types bunched at the kink.

The benefit of this perturbation comes from the efficient taxation of types above the kink and is the same as in the interior case. The optimality requires that

$$D_{a,c}^{K} = \int_c^1 N_j dj \tag{30}$$

holds, whenever types $[a, c]$ are bunched together at some positive formal income level.

Theorem 2. *The optimal allocation satisfies (28) and at every formal income level* $\bar{y}^f$

- *if there is a unique type i s.t. $y_i^f = \bar{y}^f$, then the interior conditions (25)-(27) hold for i,*
- *if there is an interval I of types such that $\forall_{i \in I} y_i^f = \bar{y}^f$, then (30) holds for I.*

Although we managed to characterize the full set of necessary optimality conditions, the interior conditions are generally easier to apply. Below we show that the interior allocation, even if not incentive-compatible, may be a good predictor of which agents will optimally work in the shadow economy.

Lemma 3. *The set of shadow workers from the interior allocation is a subset of $\mathcal{S}$ from the optimal allocation.*

3.3 Interpretation

Which agents should work in the shadow economy?

Lemma 4. *Consider a subset of population, for which $\psi_i > v'(0)$ holds. Type i optimally works in the shadow economy if*

$$\mathbb{E}\left\{1 - w_j | \, j > i\right\} \geq \frac{\phi_i - \psi_i}{\phi_i}\left(-\frac{d}{di}\left(\frac{\psi_i}{\phi_i}\right)\right)^{-1}\frac{\mu_i}{1 - M_i}. \tag{31}$$

This condition is both necessary and sufficient if the interior allocation is incentive-compatible.

In Lemma 4 we do not consider types for which $\psi_i \leq v'(0)$, since these agents will never work in the shadow economy. The consumption gain for them is always lower than the disutility cost of labor. The inequality (31) compares the gains from efficient taxation of all types above i with the cost of sending type i to the shadow economy. A type i is likely to optimally work in the shadow economy, if the planner on average puts a low marginal welfare weights on the types above i, the relative productivity loss from moving to informal employment is low and the density of distorted types is low in comparison to the fraction of types above. Finally, the shadow employment is more likely if the comparative advantage of working in the shadow sector $\frac{\psi_i}{\phi_i}$ is quickly decreasing with type. It means that higher types have less incentives to follow type i into the shadow economy.

Note that if the planner is Rawlsian and type i is higher then (or equal to) the worst-off type, then the left hand side is equal to 1 and the condition (31) is just a continuous equivalent of the condition (10) from the simple model.

The optimal tax rates Let's focus on agents that supply some formal labor and are not bunched at the kinks of the tax schedule. These types never supply informal labor. The optimal tax formula is

$$\frac{t_i}{1 - t_i} = \min\left\{\frac{\dot{\phi}_i}{\phi_i}\left(1 + \frac{1}{\zeta_i}\right)\frac{1 - M_i}{\mu_i}\mathbb{E}\left(1 - w_j | \, j > i\right), \frac{\phi_i - \psi_i}{\psi_i}\right\}. \tag{32}$$

Suppose that the productivity loss from joining the shadow economy $\frac{\phi_i - \psi_i}{\psi_i}$ is higher than the first term. Then the tax rate should be set according to the formula derived by Diamond [1998]. The expectations operator describes average social preferences towards all types above. In general, the less the planner cares about increasing utility of the types above i, the higher t_i will be. Note that if the welfare weights increase with type or G is a strictly convex function, this term may become negative, leading to negative marginal tax rates, as explained by Choné and Laroque [2010]. Since the sign of the tax rate is ambiguous, below we write how the other terms influence its absolute value. The optimal tax rate increases in absolute value when the growth rate of formal productivity with respect to type is high. If the planner is redistributive and types above i are much more productive than types below, it is optimal to set a high tax rate. The tax rate decreases with elasticity of labor supply ζ_i, since it makes the affected agents more responsive to the tax changes. The ratio $\frac{1-M_i}{\mu_i}$ tells us how many agents will be taxed in a non-distortionary manner relative to the density of distorted agents. If this ratio is high, gains from increasing tax rates will be high as well, relative to the cost.

If the Diamond formula prescribes tax rates which are too high, the optimal tax rate will equalize the return from formal and informal labor. This is the highest tax rate consistent with agents working in the formal sector.

Optimal bunching Bunching may arise at the bottom of the formal income distribution, resulting in de facto exclusion from the formal labor market. Bunching may also appear at a positive level of formal income, which implies a kink in a tax schedule. All the shadow workers are subject to bunching, yet some formal workers can be found at the kinks as well. The formal income schedule at which the kink is located is determined by

$$\frac{t_a}{t_c - t_a} = \frac{1 - M_c}{M_c - M_a}\mathbb{E}\left\{ \bar{w} - w_j | \, j \geq c \right\} - \mathbb{E}\left\{ \Delta MRS_j w_j | \, c > j \geq a \right\}, \tag{33}$$

where a and c are respectively the lowest and the highest type bunched at the kink. Note that both t_a and t_c are set according to (32). The location of the kink is determined by the trade-off between tax and welfare losses from the bunched agents and the tax revenue gains from the efficient taxation of agents above the kink.

4 Shadow and formal productivities in the data

In this section we estimate the empirical counterparts of the three key objects of the model: the formal productivity (ϕ_i), the informal productivity (ψ_i) and the distribution of types (μ_i). The estimation is conducted with data from Colombia, a country with a large shadow economy.[7] We also recover the current labor tax scheme for Colombia and use it to construct the pre-labor tax income from reported income. In section 5 we go further and analyze the differences between the current and the optimal tax scheme obtained with the estimated model.

In the model, ϕ_i and ψ_i correspond to the pre-tax (real) income for one unit of labor for individual i in each sector. In the data this corresponds to the wage that a given individual can get in each sector of the economy[8]. If the wage every individual can get in each sector were observable then it would be enough to order individuals according to their formal (pre-tax) wage, such that ϕ_i is increasing in i. With this order of individuals and the observed wages it is straightforward to compute ϕ_i, ψ_i and μ_i.

Extracting these theoretical objects from the data is difficult because workers endogenously sort themselves into each sector. This implies that we only observe the wage of a worker at one sector and; the distribution of workers across sectors depends on the wages itself. Therefore, our sample of formal and informal wages is not randomly drawn from the population, hindering the use of simple identifying assumptions to recover the productivity profiles. For example, assuming that informal productivity and formal productivity increase with type is not enough to identify the productivity

[7]58% of the workers are part of the shadow economy according to our estimates. Estimates of the size of the shadow economy by the official statistical agency in Colombia are close to ours but the definition is different as they use the scale of the firm or business to determine informality instead of the taxation criteria we use here.

[8]Or equivalently income if she is a self-account worker.

profiles, if the workers with high formal productivity tend to be in the formal sector such identification approach would lead to a positive bias on the estimated difference in productivities between sectors; a selection bias.

To solve these difficulties, we estimate a model with a factor that can explain most of the variability of wages in both sectors and use that factor to order individuals in the population. The factor we use is a linear combination of worker characteristics and job characteristics, such as the education level and the task done in the job. The estimated formal and shadow productivities correspond to the predicted wage in each sector by the factor model.

For top earners (top 0.8%), the factor cannot account for their income dispersion and the gap with respect to the rest of the population. We extend our identification strategy estimating a Pareto distribution for the wages of top earners in the formal sector. We do not need to recover in this case the informal productivity of top earners and only assume that if the single crossing property holds (ϕ_i/ψ_i is increasing in type) for non-top earners then it also holds for top earners.

We find that both productivity estimates are increasing in type and that the single-crossing property is satisfied. Specifically, the wedge between the productivity levels of each sector is almost zero for the least productive agents and formal productivity increases 47% faster in type than the minimum required by the single-crossing property. At the top the Pareto parameter is 1.8, which is close to what was found for the income distribution in the US by Saez [2001].

The main novelty of this section is that we assess the differences between the formal and the shadow economy at the worker level; controlling for the sorting of workers. Productivity as measured in Porta and Shleifer [2008] can come also from the worker characteristics and not only from the type of firms or jobs in each sector. With our approach we are able to discuss the wage differential across sectors for a given worker and job. On the other hand, the mapping of our estimates to productivity levels depends on the structure of the labor and goods market, because we rely on data on wages rather than quantities produced or profits of the firm; as those other studies do.[9] For the purposes of this paper this is not important since our object of interest is the income of the worker in each sector. For a study of the productive structure of sectors our results can only be interpreted once the link between wages and productivity is specified.[10]

The remaining of the section is organized as follows: first, we present the data used and how we identify if a worker is formal or not. Second, the empirical specification is presented and last, the results are shown and discussed.

4.1 Data

All the information we use in this section is obtained from the household survey done by the official statistical agency in Colombia (DANE). Our sample is for the year 2013 and we have 170.000

[9]Although by relying on profits or total sales data those studies have to circumvent other challenges, that come from the endogenous price adjustments to productivity changes. The most common way to solve this is documented in XXX and requires also the specification of a production function and a market structure for the goods market.

[10]For example if it is assumed perfect competition in the labor market then our measure corresponds directly to the worker marginal productivity. If a production function with constant returns to scale is also assumed then our measure also reflects the average productivity of the worker.

observations of workers. The sample includes personal information such as age, gender, years of education and also labor market related variables including hours worked, number of jobs, type of job, income sources and social security affiliation. All of the information is reported by the worker.

The variables we use from the survey can be grouped in 4 categories: worker characteristics, job characteristics, worker-firm relationship and social security status. A linear combination of the variables in the first three categories is used to construct a factor that captures the variability of wages . The fourth is used to classify workers between the formal and the shadow economy. Below we provide a brief description of the variables included in each category, for more detailed information see the Data appendix.

-*Worker characteristics* capture the type of worker irrespectively of the job he is currently doing. They include: age, gender, education level and work experience in previous jobs.

-*Job characteristics* characterize the type of job and task that the worker does irrespectively of the characteristics of the worker. The variables included are: number of workers in the firm (size), industry to which the firm belongs, geographical location of the firm and the task the worker has to do.

-*Worker-firm relationship* characterize the information about the type of contract and the wage determination. The variables included here are: The wage of the worker, number of working hours, the length of the match, if the worker is hired through an intermediary firm and if the worker belongs to a union.

-*Social security status* determine if the worker is affiliated to social security in its different dimensions and the type of affiliation. The variables included are: affiliation to the health system, the pension system and the labor accidents insurance; also who pays for the affiliation to each component.

We use the sampling weights, that correspond to each observation in the survey, in the estimation of productivities and the density of types to make our estimates representative of the Colombian economy.

Classification of workers into formal and shadow workers

Workers are not directly asked whether they belong to the formal or shadow economy, or alternatively whether they pay or not the labor income taxes. Therefore, we have to rely on indirect measures. We use the survey questions related to the compliance with the labor regulation. Specifically, the affiliation to the health security system, the pension system and the accidents insurance policy. The criteria we use to identify a formal worker is the affiliation to the three components through his work (not as a beneficiary of other worker). The affiliation status of a worker, in these three components of the social security system, is expected to be highly effective to identify which workers do not comply with labor taxes. The affiliation fees are linked administratively with the payment of the payroll tax for employees and the income tax for self account workers.

4.2 Empirical specification

The logarithm of both productivities (ϕ_i and ψ_i) can be written as a function of a single factor F_i as follows

$$\begin{aligned} \ln \phi_i &= \gamma_0^f + F_i && (34) \\ \ln \psi_i &= \gamma_0^s + \gamma_1^s F_i && (35) \end{aligned}$$

where γ_0^j, γ_1^s characterize the linear function in sector $j \in \{f, s\}$. We have set $\gamma_1^f = 1$ without loss of generality, given that this will just rescale the factor.

The factor is a linear combination of a set of n variables contained in vector X_i with weights given by the vector β. Then we have that

$$F_i = \beta X_i \quad (36)$$

The proxy we have for the model productivities are the wages of workers w_i^j in each sector j, then we have that[11]

$$\begin{aligned} \ln w_i^f &= \ln \phi_i + u_i^f && (37) \\ \ln w_i^s &= \ln \psi_i + u_i^s && (38) \end{aligned}$$

where u_i^f and u_i^s are random variables with mean zero. Wages are drawn from a probability distribution where the key location parameters are ϕ_i and ψ_i,the theoretical concepts in our analysis. In the theoretical analysis we abstract from the underlying variance of the distribution and focus on the limit when it tends to zero. The model is a static economy so we are not concerned with short term variations of wages but rather on the distribution of the location parameters across the population.

Combining equations (34) to (38) we get the specification of the empirical model that corresponds to

$$\ln w_i = \gamma_0^f + I_i \left(\gamma_0^s - \gamma_0^f \right) + \left(1 + I_i \left(\gamma_1^s - 1\right)\right) \beta X_i + u_i \quad (39)$$

where I_i is an indicator function that takes the value of 1 if type i works in the shadow economy and $u_i = I_i u_i^s + u_i^f$. We estimate 39 by non-linear least squares.

Ordering of agents and estimated productivities

Note the estimate of parameter a as $\hat{a}$. We proceed to order the individuals in our sample with indexes $i \in [0, 1]$ such that $i < i' \iff \hat{\beta} X_i < \hat{\beta} X_{i'}$. We compute the index of each individual using the following formula

$$i = \frac{\hat{\beta} X_i - min_{i'}\{\hat{\beta} X_{i'}\}}{max_{i'}\{\hat{\beta} X_{i'}\}}$$

[11] Note that, as discussed earlier, w_i^j is only observed if type i works in sector j.

that is just rescaling the factor using the minimum and the maximum values it takes in the sample. The estimated productivities of each type i then correspond to

$$\hat{\phi}_i = \exp\left\{\hat{\gamma}_0^f + \hat{\beta} X_i\right\} \tag{40}$$

$$\hat{\psi}_i = \exp\left\{\hat{\gamma}_0^s + \hat{\gamma}_1^s \hat{\beta} X_i\right\} \tag{41}$$

Single-crossing condition

The single-crossing condition states that the ratio ϕ_i/ψ_i has to be increasing in type. Using 40 and 41 this ratio can be written as

$$\frac{\hat{\phi}_i}{\hat{\psi}_i} = \frac{\exp\left\{\hat{\gamma}_0^f + \hat{\beta} X_i\right\}}{\exp\left\{\hat{\gamma}_0^s + \hat{\gamma}_1^s \hat{\beta} X_i\right\}}$$

$$\Longrightarrow \frac{\hat{\phi}_i}{\hat{\psi}_i} = \exp\left\{\hat{\gamma}_0^f - \hat{\gamma}_0^s\right\} \exp\left\{\left(1 - \hat{\gamma}_1^s\right) \hat{\beta} X_i\right\}$$

Then, if $\hat{\gamma}_1^s < 1$ we have that the single-crossing condition is satisfied. Recall that we standardized to 1 the marginal (percentile) increase of formal productivity to a marginal increase in the factor. Therefore, this condition states that a marginal increase in the factor has to imply a lower marginal increase in shadow than in formal productivity.

Top income earners

Note that since w_i is in units of year income for full time work, then $\hat{\phi}_i$ corresponds (on average) to the maximum income that type i can achieve. Nevertheless, some income observations are above the maximum value implied by the factor for the most productive worker working full time.. That is, there could be yearly labor income observations y_i that satisfy

$$y_i > max_{i\prime}\{\hat{\phi}_{i\prime}\} = \hat{\phi}_1 \tag{42}$$

We classify the individuals that satisfy this criterion as top earners. These are individuals with a very large wage premium that cannot be accounted for with our benchmark specification and for which their wage does not seem to have the same relationship with the factor as the rest of the population.

To characterize with more accuracy this behavior at the top of the income distribution we estimate the upper tail of the productivity distribution by fitting a Type I Pareto distribution for the gross wage w of top earners. The support of the distribution is given by $\left[\hat{\phi}_1, \infty\right)$ and the shape parameter is estimated by maximum likelihood.

A final adjustment has to be made to the index of agents, to fit the top earners in the type space $[0, 1]$ we compress the indexes on non-top earners to the interval $[0, k]$ and top earners are assigned to $[k, 1]$ and ordered by their gross wage.

Distribution of types

The assignment of indexes for each observation and their corresponding sampling weights implies a discrete distribution of workers (non-top earners). The continuous distribution of types is obtained by a kernel density estimation with a linear interpolation at the evaluation points. The estimated kernel distribution gives us the distribution of types in the interval $[0, k]$.

There are many (potentially a continuum) probability distributions and formal productivity profiles for the types in the interval $[k, 1]$ (top earners). For top earners we have a Pareto distribution for productivities with the support $[max_{i'}\{\hat{\phi}_{i'}\}, \infty)$ but this distribution can be replicated by different types distributions in $[k, 1]$ at the types space, provided that the formal productivities ϕ_i for $i \in [k, 1]$ are adjusted accordingly. This phenomenon does not occur with non-top earners because their productivity profiles are given by our parametric model.

There are two requirements that the distribution of types and productivity profiles of top earners satisfy always: the total mass of the distribution has to coincide with the mass of top earners and that $\lim_{i\to 1} \phi_i = \infty$.

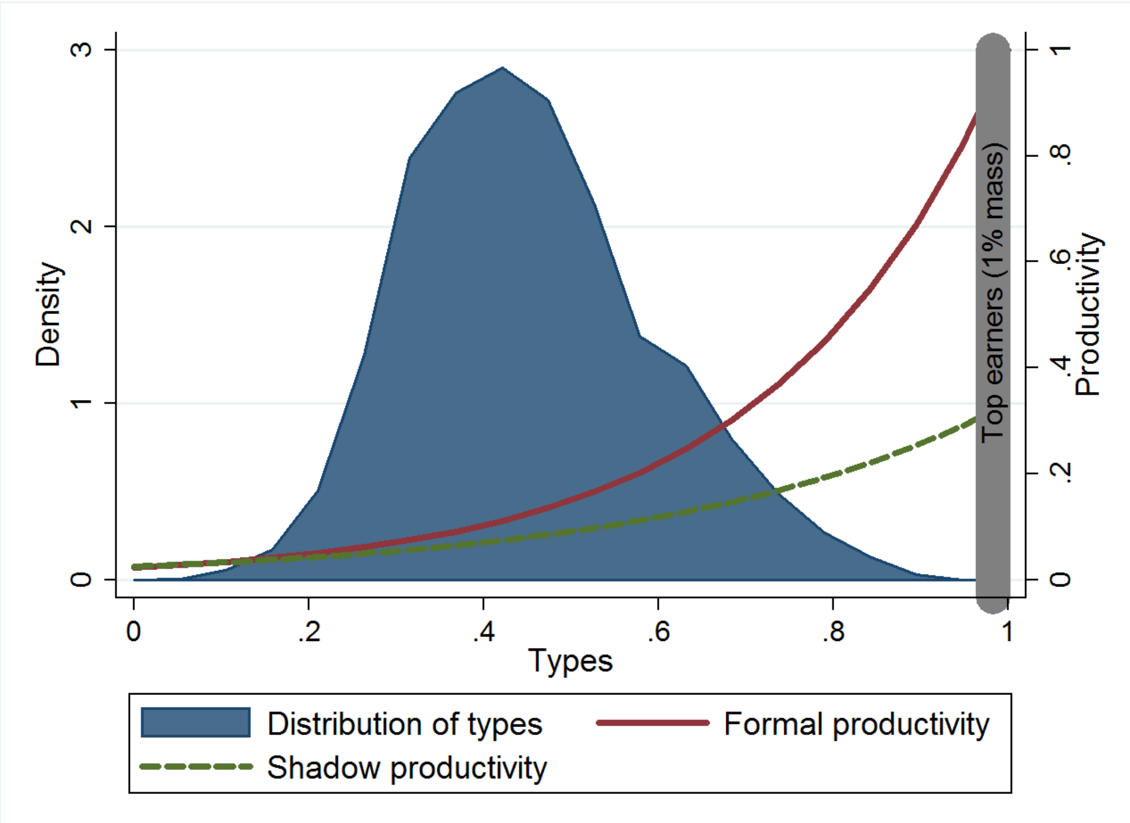
4.3 Estimation results

Here we discuss the results of the estimation of the formal productivity (ϕ_i), the informal productivity (ψ_i) and the distribution of types (μ_i). Parameter estimates for β and the detailed description of the variables included in X_i are presented in the appendix.

Figure (3) presents the estimated productivities and the types distribution for non-top earners. The estimated values of γ_0^f and γ_0^s are almost identical with $\hat{\gamma}_0^s$ slightly greater so type 0 is slightly more productive in the shadow economy. The single-crossing condition is supported by the data since the hypothesis $\gamma_1^s < 1$ is not rejected at a 1% confidence level. The most productive individual among non-top earners is almost three times more productive in the formal economy than in the shadow economy.

Top earners are assigned to the set $[0.98, 1]$, the estimated value of the shape parameter of the Pareto distribution is 1.81 and comprise a mass of about 1% of the total population (details of the estimation are presented in the appendix). The shaded region in Figure (3) corresponds to the top earners. We do not plot their productivity profiles and density. Recall that what is identified is the distribution of formal productivities at the top with support $[max_{i'}\{\hat{\phi}_{i'}\}, \infty)$ and this can be matched with many different combinations of formal productivity and probability density specifications in the types space; all of them equivalent for the optimal taxation problem that solves the planner. The shadow productivity for top earners is not specified, the only condition imposed on it is that the single crossing property has to hold also at the top.

Figure 3: Estimated productivities and types distributions



5 Calibrated exercise [preliminary]

We assume that the agents' utility function is

$$U(c,n) = c - \Gamma \frac{n^{1+\gamma}}{1+\gamma},$$

where we restrict n to lay in the interval $[0,1]$. We set the elasticity of labor supply $\frac{1}{\gamma}$ to $\frac{1}{2}$, which is a relatively high value in the literature. In this way we take a conservative stance on the amount of redistribution the planner can conduct. We set Γ to 1.64 in order to match the average hours worked in the formal economy.

We simulate the model economy with the actual, current tax schedule of Colombia. Table 1 shows that matching the productivity distribution and the income tax schedule alone is enough to generate a shadow economy similar to the one observed in the data. Nevertheless, there are some discrepancies. Our model economy has a higher proportion of informal workers and lower proportion of informal income than the data shows. It seems that the model predicts a sharper partition of types according to the productivity: agents with low productivity work in the shadow economy, while the rest work formally. The data, however, is more less clear-cut and contains instances of people that are quite productive, yet work in the shadow economy.

Table 1: Validation moments

	Data	Model
share of workers in the shadow economy	57.99%	66.02%
share of shadow income	30.94%	18.95%

5.1 Comparison of the actual and the optimal allocation

We find the optimum according to the Rawlsian welfare criterion, i.e. the planner maximizes the transfer given to the poorest. We require that the planner obtains the same net tax revenue as the actual tax schedule.

Table 2 compares the key moments of the optimal and the actual allocation. Optimally, the shadow economy should employ 30% of the Colombian work force, which is roughly a half of the actual employment. The share of shadow income falls even more, since the optimal tax makes mostly the least productive workers to work in the shadow economy. The implementation of the optimal tax allows the planner to increase the transfer to the poorest by 50%, which results in a more than 30% increase in welfare (in consumption terms) in comparison to the actual tax schedule. Although more people work in the productive formal sector, the average labor income falls. The marginal tax rates required to finance the redistribution are high and discourage labor supply to the extend that the total output is reduced.[12]

Table 2: The actual and the optimal allocation

	Actual	Optimal
welfare (consumption equivalent)	100%	131.44%
average labor income (formal + shadow)	100%	98.91%
share of workers in the shadow economy	66.02%	30.67%
share of shadow income	18.95%	5.24%

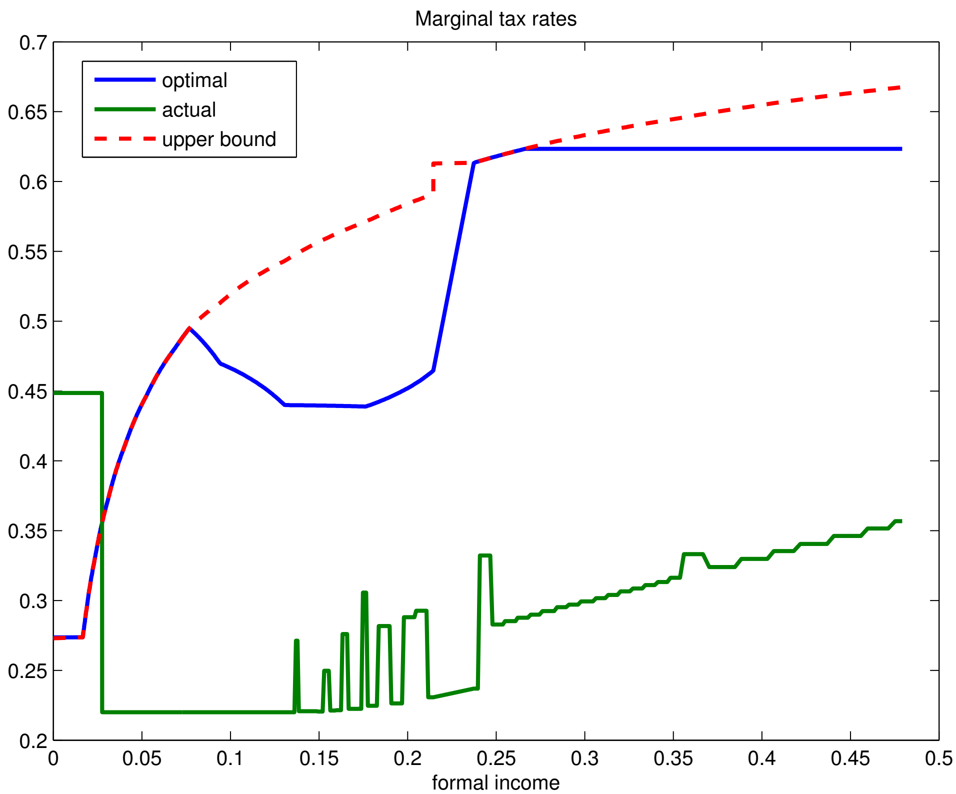
In Figure 4 we can see the marginal tax rates in the two regimes. The actual tax schedule (the green curve) involves high 45% marginal rate at low levels of income, implied by phasing-out of transfers. As income increases the rate drops to 22% and remains flat - workers with this income pay only the flat payroll tax. The progressive income tax starts at higher income levels and gradually increases the marginal tax up to 49%. This tax is a step function with over 80 steps of varying width. The graphs represents the marginal tax rate implied by the linear interpolation of the steps.[13]

In comparison to the actual tax rate, the optimal tax rate (the blue curve in Figure 4) is lower at low levels of income and much higher elsewhere. Lower marginal taxes at the bottom mean transfers are phased-out more slowly, so less productive workers have less incentives to move to the informal sector. Higher marginal tax rates elsewhere imply that the richest agents pay much higher total tax than in the actual economy, which allows the planner to finance the generous transfer. Recall that by (32) the marginal tax rate is set according to the Diamond [1998] formula unless it violates

[12] The marginal tax rates chosen by the Ralwsian planner maximize the tax revenue. Hence, the Rawlsian optimum is exactly at the top of the Laffer curve.

[13] The true tax involves 0 rate at the interior of each step and an unbounded rate between steps, hence it cannot be represented on such graph.

Figure 4: Comparison of the actual and the optimal tax schedule



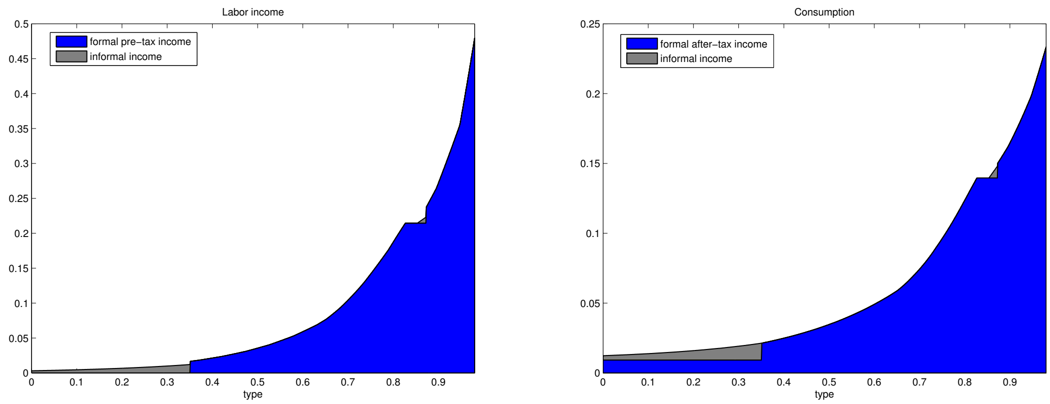
the upper bound imposed by the shadow economy. This upper bound (the red dashed curve in Figure 4) stands for the highest marginal tax rate that is consistent with agents at this income level working in the formal economy. Note that for many agents the marginal tax rate lays at the upper bound, which means that it is set at the different level than the standard Mirrlees model implies. Specifically, the tax schedule implied by the standard Mirrlees model without shadow economy considerations has a much higher marginal tax rate at the bottom. Such tax would lead to a much larger informal sector in comparison to to the optimal one.

The optimal tax schedule involves a kink at the income level of 0.22. It is visible clearly in Figure 5, showing the optimal allocation of income and consumption by types. The discontinuous increase of the tax rate is optimal since the density of types falls very rapidly. Note that some of the agents at the kink supply shadow labor. Hence, it is optimal to allow some highly productive types to work part-time in the shadow economy in order to efficiently tax the even more productive agents.

6 Conclusions

A large fraction of the economic activity in most countries is carried out in the shadow sector. Our analysis shows that this fact pose an additional challenge for the design of the tax policy. The

Figure 5: The Rawlsian optimum



existence of a shadow economy generates new tradeoffs in the optimal design of the tax system, as agents can move across sectors in response to a tax policy change.

The shadow economy can limit redistribution if the difference in productivities across sectors is low, in particular for the highly productive agents. If this is not the case, we show that the shadow economy can be used as an additional instrument of redistribution and may lead to an increase in welfare. The optimal tax scheme in this case is characterized by the high marginal tax rates at the low levels of income which increase the revenues from the workers with high productivity. Such a policy can improve welfare, since the high marginal tax at low levels of reported income does not distort the labor decision of the less productive workers if they work in the shadow economy.

Three objects are key to determine the optimal size of the shadow economy from our optimal taxation perspective: formal productivity, the informal productivity and the probability distribution of workers at the different productivity levels. The mechanism proposed has a quantitatively sizable effect. In the case of Colombia we find that the analysis provides a rationale for a large shadow economy. Nevertheless, the observed levels of informality are even higher and double the optimal level.

With respect to the social security system, this paper suggests that, under some circumstances, allowing less productive people to collect welfare benefits and simultaneously work in the shadow economy is desirable. Furthermore, policies that are designed to deter the creation of shadow jobs should aim at those that are taken by very productive agents in the formal sector. It is important to stress that the way the shadow economy is modeled in this paper abstracts from many issues, such as competition between formal and informal firms, lack of regulation and law enforcement, as well as potential negative externalities caused by the informal activity. All those phenomena are likely to reduce the potential welfare gains from exploiting the shadow economy.

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Appendix

A Proofs from Section 2

Proposition 2.

Proof. Note that the first-best allocation is consistent with the additional constraint (5), hence it is the solution to the planner's problem. Essentially, conditional on truthfully revealing type, incentives of the agent and the planner regarding the shadow labor are perfectly aligned. If a given type pays taxes according to the true type, choosing shadow labor in order to maximize utility cannot hurt the social welfare.

□

Proposition 3.

Proof. Existence follows from the Weierstrass theorem. The objective function is continuous. All constraints consist of equalities or weak inequalities, so the choice set is a closed set. We can ensure boundedness of the choice set by considering the allocations that yield welfare not lower than the allocation without transfers, *laissez-faire*. Since *laissez-faire* satisfies all the constraints, the optimum cannot be worse. Note the welfare of laissez-faire by $W_{lf} > -\infty$. Suppose that the set of allocations with welfare greater than W_{lf} is unbounded. By the resource constraint, it means that the labor supply of some type in some allocations has to diverge to infinity. By $\lim_{n\to\infty} v'(n) = \infty$, it implies that disutility from labor of this type, as well as welfare of the whole allocation, goes to $-\infty$. It contradicts the assumption that the allocation is superior than *laissez-faire*. Finally, notice that our restricted choice set is always non-empty, since it contains *laissez-faire*.

In the first-best, $U(c_l, n_l) \geq U(c_h, n_h)$. By assumption of $v'(0) = 0$, we know that $n_l^f > 0$. Then the utility of h mimicking l is $U\left(c_l, \frac{\phi_l}{\phi_h} n_l^f\right) > U\left(c_l, n_l^f\right) \geq U(c_h, n_h)$, which violates $IC_{h,l}$. Hence, the optimum is not the first-best.

Suppose that at the optimum $IC_{h,l}$ does not bind. First, let's consider the case in which $U(c_h, n_h) > U(c_l, n_l)$. Since $IC_{h,l}$ is slack, the planner may increase transfers from h to l, which raises welfare, so it could not be the optimum in the first place. Second, suppose that $U(c_l, n_l) \geq U(c_h, n_h)$. It can happen only if $n_l^s > 0$. Otherwise, as we have shown above, $IC_{h,l}$ is violated. If $n_l^s > 0$ and $IC_{h,l}$ is slack, the planner can marginally decrease n_l^s and increase n_l^f, which generates free resources. Hence, at the optimum $IC_{h,l}$ has to bind.

Suppose that $IC_{l,h}$ binds. If the resource constraint is satisfied as equality, it may happen only if l type is paying a positive tax, while h type receives a transfer. Then the planner can improve welfare by canceling the redistribution altogether and reverting to *laissez-fare*, where none of the incentive constraints bind.

$\square$

Lemma 5. *At the optimum either $U(c_l, n_l) = U(c_h, n_h)$ and $n_l^s > 0$, or the following optimality condition holds*

$$\min\left\{\frac{v'(n_l)}{\phi_l} - \left(\mu_l + \mu_h \frac{v'(n_{h,l})}{\phi_h}\right), n_l^f\right\} = 0, \tag{43}$$

where $n_{i,-i} = \frac{\phi_{-i}}{\phi_i} n_{-i}^f + n_i^s \left(\frac{\phi_{-i}}{\phi_i} n_{-i}^f\right)$ is the total labor supply of type i pretending to be of type $-i$. Suppose that v'' is nondecreasing. If $\frac{\phi_h}{\phi_l} g(\psi_h) \geq g(\psi_l)$ then this optimality condition is sufficient for the optimum.

Proof. If $U(c_l, n_l) = U(c_h, n_h)$ and $n_l^s = 0$, then such allocation is not incentive compatible. The proof is identical as the proof of the claim that the first-best is not incentive compatible in Proposition 3. Hence, if $U(c_l, n_l) = U(c_h, n_h)$, then $n_l^s > 0$.

Let's consider the case in which $U(c_h, n_h)$ is always greater than $U(c_l, n_l)$. $IC_{h,l}$ has to bind, otherwise the planner could equalize utilities of both types. Consider changing n_l^f by a small amount and adjusting T_l such that $IC_{h,l}$ is satisfied as equality. It means that

$$\frac{dT_l}{dn_l^f} = \phi_l \mu_h \left(1 - \frac{v'(n_{h,l})}{\phi_h}\right).$$

This perturbation affects social welfare by

$$\frac{dU(c_l, n_l)}{dn_l^f} = \phi_l - \frac{\partial T_l}{\partial n_l^f} - v'(n_l) = \phi_l \left(\mu_l + \mu_h \frac{v'(n_{h,l})}{\phi_h}\right) - v'(n_l).$$

Optimum requires that either $\frac{dU(c_l,n_l)}{dn_l^f} = 0$ or $\frac{dU(c_l,n_l)}{dn_l^f} < 0$ and $n_l^f = 0$, which results in (43). Sufficiency of this first order condition depends on the second order derivative of welfare with respect to the perturbation. In order to have the second derivative well behaved, we are going to assume that v'' is nondecreasing. Then, we need to consider two cases (see Table 3). If $\frac{\phi_h}{\phi_l} g(\psi_h) \geq g(\psi_l)$ holds, then $\frac{dU(c_l,n_l)}{dn_l^f}$ is non-increasing in n_l^f. It means that

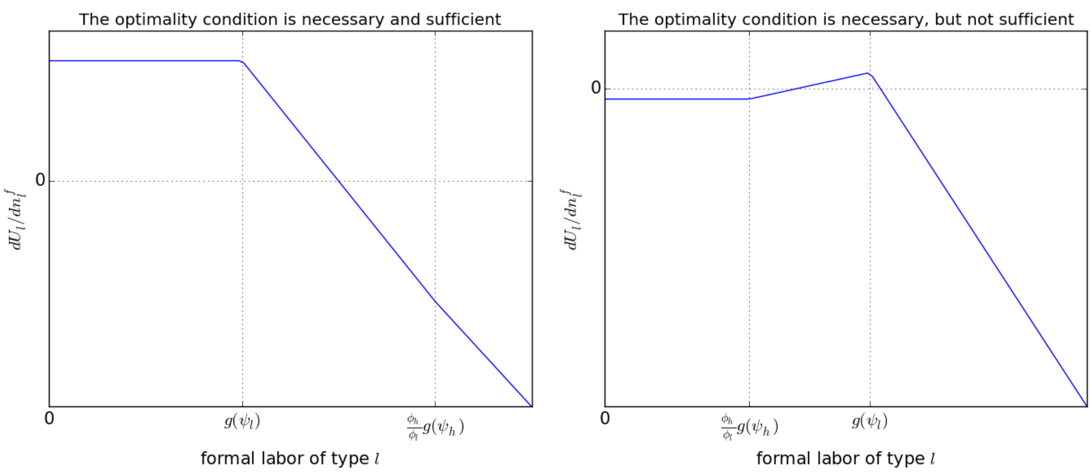
Table 3: Second order derivative of welfare with respect to the perturbation

The case of $\frac{\phi_h}{\phi_l} g(\psi_h) \geq g(\psi_l)$			
	$n_l^f < g(\psi_l)$	$g(\psi_l) < n_l^f < \frac{\phi_h}{\phi_l} g(\psi_h)$	$\frac{\phi_h}{\phi_l} g(\psi_h) < n_l^f$
$\frac{d^2 U(c_l,n_l)}{dn_l^{f2}} =$	0	$-v''\left(n_l^f\right) < 0$	$\mu_h \left(\frac{\phi_l}{\phi_h}\right)^2 v''\left(\frac{\phi_l}{\phi_h} n_l^f\right) - v''\left(n_l^f\right) < 0$
The case of $\frac{\phi_h}{\phi_l} g(\psi_h) < g(\psi_l)$			
	$n_l^f < \frac{\phi_h}{\phi_l} g(\psi_h)$	$\frac{\phi_h}{\phi_l} g(\psi_h) < n_l^f < g(\psi_l)$	$g(\psi_l) < n_l^f$
$\frac{d^2 U(c_l,n_l)}{dn_l^{f2}} =$	0	$\mu_h \left(\frac{\phi_l}{\phi_h}\right)^2 v''\left(\frac{\phi_l}{\phi_h} n_l^f\right) > 0$	$\mu_h \left(\frac{\phi_l}{\phi_h}\right)^2 v''\left(\frac{\phi_l}{\phi_h} n_l^f\right) - v''\left(n_l^f\right) < 0$

the optimality condition (43) is sufficient. If $\frac{\phi_h}{\phi_l} g(\psi_h) < g(\psi_l)$, then $\frac{dU(c_l,n_l)}{dn_l^f}$ is not monotone in n_l^f and it may be the case that (43) points at either local maximum which is not a global maximum, or at the local minimum.

Figure 6 shows these two cases. In the first panel $\frac{\phi_h}{\phi_l} g(\psi_h) \geq g(\psi_l)$ holds and the optimality condition (43) always points at the optimum (in this case, the value of n_l^f where $\frac{dU(c_l,n_l)}{dn_l^f} = 0$). In the second panel $\frac{\phi_h}{\phi_l} g(\psi_h) < g(\psi_l)$ holds and the optimality condition is not sufficient. There are three points that satisfy condition (43): local maximum at $n_l^f = 0$, local minimum with $n_l^f \in \left(\frac{\phi_h}{\phi_l} g(\psi_h), g(\psi_l)\right)$ and the other local maximum with $n_l^f > g(\psi_l)$.

Figure 6: Sufficiency of the optimality condition



□

Proposition 4.

Proof. In the proof of Lemma 5 above we described the impact of changing formal labor of l on the social welfare, $\frac{dU(c_l,n_l)}{dn_l^f}$. The condition (10) describes situations when the impact of the perturbation is non-positive at $n_l^f = 0$. From Figure 6 it is clear that if it is not the case, type l will never optimally work in the shadow economy.

Suppose that $\frac{\phi_h}{\phi_l} g(\psi_h) \geq g(\psi_l)$. Condition (10) implies that $\frac{dU(c_l,n_l)}{dn_l^f}$ is always non-positive, so it is optimal to reduce n_l^f as long as $U(c_h, n_h) > U(c_l, n_l)$. From Lemma 5 we know also that $U(c_h, n_h) > U(c_l, n_l)$ if l works only formally, so it is optimal to place type l in the shadow economy.

Now suppose that $\frac{\phi_h}{\phi_l} g(\psi_h) < g(\psi_l)$. Condition (11) means that the maximum of $\frac{dU(c_l,n_l)}{dn_l^f}$ attained at $n_l^f = g(\psi_l)$ (see Figure 6) is non-positive. Therefore, it is optimal to reduce n_l^f until utilities of both types are equalized, which can happen only when l works in the shadow economy. Condition (11) is sufficient, but not necessary for l to work in the shadow economy, because the social welfare changes in a non-monotone way with n_l^f. If (11) is not satisfied, marginally increasing n_l^s from 0 is bad for welfare, but increasing it further may eventually lead to welfare gains, and the total effect on welfare is ambiguous.

□

Proposition 5

Proof. Suppose that optimally $n_l^s > 0$. From Figure 6 it is clear that in such situation it is in the best interest of type l to work exclusively in the shadow economy. However, if $\psi_l > \psi_h$ and $n_l^f = 0$, the incentive compatibility constraint of the type h implies that

$$U(c_l, n_l) = U(\psi_l n_l^s - T_l, n_l^s) > U\left(\psi_h n_{h,l}^s - T_l, n_{h,l}^s\right) = U(c_h, n_h).$$

Since the planner is Rawlsian, such allocation is not desirable. The planner will rather stop decreasing n_l^f at the point where utilities of both types are equal. On the other hand, if $\psi_l \leq \psi_h$ then

$$U(c_l, n_l) = U(\psi_l n_l^s - T_l, n_l^s) \leq U\left(\psi_h n_{h,l}^s - T_l, n_{h,l}^s\right) = U(c_h, n_h),$$

so the planner will optimally decrease n_l^f to zero.

□

Proposition 6.

Proof. In order to examine when the optimum welfare is strictly higher than in the standard Mirrlees model, we will compare utility of type l in the standard Mirrlees model $\left(U\left(c_l^M, n_l^M\right)\right)$ and in the shadow economy model, when l is working only in the shadow economy $\left(U\left(c_l^{SE}, n_l^{SE}\right)\right)$. Clearly, when the second scenario yields higher utility, the existence of the shadow economy is welfare improving.

In the standard Mirrlees model, the binding constraint is $U\left(\phi_h n_h^M, n_h^M\right) - T_h^M = U\left(\phi_l n_l^M, \frac{\phi_l}{\phi_h} n_l^M\right) - T_l^M$. The resource constraint implies that $T_h^M = -\frac{\mu_l}{\mu_h} T_l^M$ and we get $T_l^M = \mu_h \left(U\left(\phi_l n_l^M, \frac{\phi_l}{\phi_h} n_l^M\right) - U\left(\phi_h n_h^M, n_h^M\right)\right)$. Now, the utility of type l is

$$U\left(c_l^M, n_l^M\right) = U\left(\phi_l n_l^M, n_l^M\right) - T_l = U\left(\phi_l n_l^M, n_l^M\right) - \mu_h \left(U\left(\phi_l n_l^M, \frac{\phi_l}{\phi_h} n_l^M\right) - U\left(\phi_h n_h^M, n_h^M\right)\right).$$

Using the same steps, we can express the utility of type l working only in the shadow economy as

$$U\left(c_l^{SE}, n_l^{SE}\right) = U\left(\psi_l n_l^{SE}, n_l^{SE}\right) - \mu_h \left(U\left(\psi_h n_{h,l}^{SE}, n_{h,l}^{SE}\right) - U\left(\phi_h n_h^{SE}, n_h^{SE}\right)\right).$$

Since there are no distortions at the top and no wealth effects, $n_h^M = n_h^{SE}$. The shadow economy is welfare improving if

$$U\left(c_l^{SE}, n_l^{SE}\right) - U\left(c_l^M, n_l^M\right) = U\left(\psi_l n_l^{SE}, n_l^{SE}\right) - U\left(\phi_l n_l^M, n_l^M\right) + \mu_h \left(U\left(\phi_l n_l^M, \frac{\phi_l}{\phi_h} n_l^M\right) - U\left(\psi_h n_{h,l}^{SE}, n_{h,l}^{SE}\right)\right) > 0.$$

The first difference (the efficiency gain) is positive if $\psi_l > \bar{\psi}_l$. The second difference (the redistribution gain) is positive when $\psi_h < \bar{\psi}_h$. Hence, when both inequalities hold weakly and at least one holds strictly, the existence of the shadow economy improves welfare in comparison to the standard Mirrlees model.

Now we will show that when the inequalities hold in the other direction, the shadow economy hurts welfare. Suppose that $\psi_l = \bar{\psi}_l$ and $\psi_h = \bar{\psi}_h$. We will prove that allocation ^{SE} is a unique optimum at this point. First we will show that when the redistribution gain is non-positive, it is true that $n_h^{SE} > \frac{\phi_l}{\phi_h} n_l^M$. Suppose on the contrary that $n_h^{SE} \leq \frac{\phi_l}{\phi_h} n_l^M$. Then we can write the following sequence of inequalities

$$U\left(\phi_l n_l^M, \frac{\phi_l}{\phi_h} n_l^M\right) \geq U\left(\phi_h n_h^{SE}, n_h^{SE}\right) > U\left(\psi_h n_h^{SE}, n_h^{SE}\right).$$

The first inequality comes from the fact that $\frac{\phi_l}{\phi_h} n_l^M$ is below the efficient level of labor supply of type h, so lowering the labor of this type even further to n_h^{SE} will decrease the utility. The second inequality is simply implied by our assumption $\phi_h > \psi_h$. This sequence of inequalities implies that the redistribution gain is strictly positive. Hence, if the redistribution gain is non-positive, $n_h^{SE} > \frac{\phi_l}{\phi_h} n_l^M$ holds.

Note that $n_h^{SE} > \frac{\phi_l}{\phi_h} n_l^M$ means that the optimal allocation of the standard Mirrlees model is not incentive-compatible with the shadow economy - deviating type h would supply some additional shadow labor. Hence, any allocation which yields the social welfare equal or higher than $U\left(c_l^M, n_l^M\right)$ has to involve type l working in the shadow economy.

Let's go back to the optimal allocation with the shadow economy, when $\psi_l = \bar{\psi}_l$ and $\psi_h = \bar{\psi}_h$. From the considerations above we know that the optimum involves some shadow labor. If we sum the efficiency gain and the redistribution gain divided by μ_h and rearrange the terms, we get

$$\left(U\left(\phi_l n_l^M, \frac{\phi_l}{\phi_h} n_l^M\right) - U\left(\phi_l n_l^M, n_l^M\right)\right) - \left(U\left(\psi_h n_h^{SE}, n_h^{SE}\right) - U\left(\psi_l n_l^{SE}, n_l^{SE}\right)\right) = 0.$$

The expression in the first brackets is positive. Hence, the second brackets are positive as well, which means that $\psi_h > \psi_l$. By Proposition 5 type l will work exclusively in the shadow economy.

To sum up, we know that at $(\psi_l, \psi_h) = \left(\bar{\psi}_l, \bar{\psi}_h\right)$ the optimum of the shadow economy model is unique and involves type l working entirely in the shadow economy. Consequently, a decrease in the shadow productivity of type l or an increase in the shadow productivity of type h leads to a strict welfare loss, since it either decreases the effective productivity of type l or decreases the transfer type l receives.

□

Proposition 7

Proof. Suppose that $\lambda_i \leq \lambda_{-i}$. In this case the $IC_{i,-i}$ may bind (it will if the inequality is strict), while $IC_{-i,i}$ is always slack. The planner will not distort the allocation of type i. Without distortions, this type will never work in the shadow economy.

Suppose that $\lambda_i > \lambda_{-i}$, so that $IC_{-i,i}$ binds. Perturb n_i^f and adjust T_i such that $IC_{-i,i}$ holds as equality:

$$\frac{dT_i}{dn_i^f} = \phi_i \mu_{-i}\left(1 - \frac{v'\left(n_{-i,i}\right)}{\phi_{-i}}\right).$$

This perturbation affects social welfare by

$$\begin{aligned} \frac{dW}{dn_i^f} &= \lambda_i \mu_i \left(\phi_i - \frac{\partial T_i}{\partial n_i^f} - v'\left(n_i\right)\right) + \lambda_{-i}\mu_{-i}\frac{\mu_i}{\mu_{-i}}\frac{\partial T_i}{\partial n_i^f} \\ &= \lambda_i \mu_i \phi_i \left(\left(1 - \frac{v'\left(n_i\right)}{\phi_i}\right) + \left(\frac{\lambda_{-i}}{\lambda_i} - 1\right)\mu_{-i}\left(1 - \frac{v'\left(n_{-i,i}\right)}{\phi_{-i}}\right)\right) \end{aligned} \tag{44}$$

Suppose that $\frac{\psi_{-i}}{\phi_{-i}} \geq \frac{\psi_i}{\phi_i}$ and $n_i^f \leq g\left(\psi_i\right)$, which means that $v'\left(n_i\right) = \psi_i$. Note that $\frac{v'\left(n_{-i,i}\right)}{\phi_{-i}} \geq \frac{\psi_{-i}}{\phi_{-i}} \geq \frac{\psi_i}{\phi_i}$. Hence

$$1 - \frac{\psi_i}{\phi_i} \geq 1 - \frac{v'\left(n_{-i,i}\right)}{\phi_{-i}} > \left(1 - \frac{\lambda_{-i}}{\lambda_i}\right)\mu_{-i}\left(1 - \frac{v'\left(n_{-i,i}\right)}{\phi_{-i}}\right),$$

which means that $\frac{dW}{dn_i^f} > 0$. Therefore, it is never optimal to decrease n_i^f so much that type i works in the shadow economy.

□

Proposition 8

Proof. First we will show how to obtain (14). The efficiency gain is straightforward. In order to obtain the redistribution gain, note that there are no distortions imposed on type $-i$, hence

$$\mu_{-i}\lambda_{-i}\left(U\left(c_{-i}^{SE},n_{-i}^{SE}\right)-U\left(c_{-i}^{M},n_{-i}^{M}\right)\right)=\mu_{-i}\lambda_{-i}\left(T_{-i}^{M}-T_{-i}^{SE}\right)=-\mu_{i}\lambda_{-i}\left(T_{i}^{M}-T_{i}^{SE}\right).$$

Summing up the terms results in (14). In order to derive thresholds, recall that $H(\psi)=U(\psi g(\psi),g(\psi))$. The efficiency gain is given by

$$\mu_i\lambda_i\left(H(\psi_i)-U\left(\phi_i n_i^M,n_i^M\right)\right),$$

it is strictly increasing in ψ_i and positive for $\psi_i>\bar{\psi}_i$. Note that by (44) n_i^M will always be distorted (downwards if $i=l$, upwards if $i=h$). Hence, $U\left(\phi_i n_i^M,n_i^M\right)<H(\phi_i)$ and the threshold $\bar{\psi}_i$ is strictly lower than ϕ_i.

Using the binding $IC_{-i,i}$ constraint, we can express the redistribution gain as

$$\mu_i\mu_{-i}\left(\lambda_i-\lambda_{-i}\right)\left(U\left(\phi_i n_i^M,\frac{\phi_i}{\phi_{-i}}n_i^M\right)-H(\psi_{-i})\right).$$

It is strictly decreasing in ψ_{-i} and is positive for $\psi_{-i}<\bar{\psi}_{-i}$. Since $\frac{\phi_i}{\phi_{-i}}n_i^M\neq g(\phi_{-i})$, it is true that $U\left(\phi_i n_i^M,\frac{\phi_i}{\phi_{-i}}n_i^M\right)<H(\phi_{-i})$ and the threshold $\bar{\psi}_{-i}$ is strictly lower than ϕ_{-i}.

□

B Proofs from Section 3

Lemma 1.

Proof. The single-crossing requires that $\frac{\partial}{\partial i}\left(\frac{V_y\left(y^f,T,i\right)}{V_T\left(y^f,T,i\right)}\right)<0$. Suppose that $v'\left(\frac{y^f}{\phi_i}\right)<\psi_i$. Then the agent supplies no informal labor and the indirect utility function V is just U evaluated at the given allocation. Since v' is increasing, the single crossing is holds. When $v'\left(\frac{y^f}{\phi_i}\right)\geq\psi_i$, then the optimal provision of informal labor means that $v'(n_i)=\psi_i$, which implies $\frac{V_y}{V_T}=\frac{\psi_t}{\phi_t}$. Therefore the single crossing condition requires that $\frac{d}{dt}\left(\frac{\psi_t}{\phi_t}\right)<0$.

□

Proposition 9.

Proof. The proof follows from the analogous proof in Doligalski [2015] (Proposition 1) by simply replacing the income schedule y with the formal income schedule y^f. The only significant difference comes in the definition of $\dot{V}_i$. In order to show that if $y_a^f>y_b^f$, then $\dot{V}_i\left(y_a^f,T_a\right)>\dot{V}_i\left(y_b^f,T_b\right)$, note that we can write $\dot{V}$ as

$$\dot{V}_i\left(y^f,T\right)=\left(\frac{\dot{\phi}_i}{\phi_i}\frac{y^f}{\phi_i}+\frac{\dot{\psi}_i}{\psi_i}\max\left\{g(\psi_i)-\frac{y^f}{\phi_i},0\right\}\right)v'(n_i).$$

The single-crossing implies that $\frac{\dot{\phi}_i}{\phi_i}>\frac{\dot{\psi}_i}{\psi_i}$, so $\dot{V}_i\left(y^f,T\right)$ is increasing in y^f.

□

Theorem 1.

Proof. First we will derive formally the term D_i^s. Then we will show that conditions from the theorem are necessary. Finally we will prove sufficiency. The interior optimality conditions for the formal type are derived in Doligalski [2015].

Suppose that $i \in \mathcal{S}$. The individual rationality implies that $v'(n_i) = \psi_i \implies n_i = g(\psi_i)$, where g is an inverse function of v'. The marginal information rent can be expressed as

$$\dot{V}_i = \left(\frac{\dot{\phi}_i}{\phi_i} n_i^f + \frac{\dot{\psi}_i}{\psi_i} \left(g(\psi_i) - n_i^f \right) \right) \psi_i. \tag{45}$$

We marginally perturb the formal income of i. In order to keep the utility schedule continuous at i, the perturbation has to be accompanied by a change in total tax such that the utility of this type is unchanged. The required change of the tax paid is $dT = 1 + \frac{dV_i}{dy^f}$, which for the shadow worker equals $\frac{\phi_i - \psi_i}{\phi_i}$. By multiplying the tax revenue change with μ_i and normalizing it with $\frac{d\dot{V}_i}{dy^f}$, we obtain the tax revenue cost of decreasing the marginal information rent of type i, given by (24).

If the formal income is nondecreasing in type, the allocation implied by the conditions (25), (26), (27) and (28) is incentive-compatible. The necessity of these conditions was demonstrated in the main text. They are sufficient when the cost of decreasing the marginal information rent of a given type is nondecreasing. It happens when for each type ζ_i is non-increasing in the labor supply (then D_i^f is nondecreasing in the formal labor supply) and when $\frac{\dot{\phi}_i}{\phi_i} + \frac{\dot{\psi}_i}{\psi_i}\zeta_i \geq 0$ holds (then for each type $D_i^f \leq D_i^s$).

□

Lemma 2.

Proof. Suppose set $\mathcal{S}$ is nonempty. If $\mathcal{S}$ either not convex or not at the bottom of the type space, then there is a left limit point of some subset of $\mathcal{S}$, denoted by i, which does not belong to $\mathcal{S}$. Since i is to the left of some subset of $\mathcal{S}$, there is type $j > i$ which belongs to $\mathcal{S}$. But $y_i^f > y_j^f = 0$, so the formal income is decreasing.

For any type $i \in \mathcal{M}$ it is true that $v'\left(n_i^f\right) = \psi_i$. Denote the inverse function of v' by g, then $n_i^f = g(\psi_i)$. The derivative of formal income with respect to type is

$$\dot{y}_i^f = \frac{d\phi_i g(\psi_i)}{di} = \dot{\phi}_i g(\psi_i) + \phi_i \dot{\psi}_i g'(\psi_i) = \frac{1}{\phi_i g(\psi_i)} \left(\frac{\dot{\phi}_i}{\phi_i} + \frac{\dot{\psi}_i}{\psi_i} \frac{\psi_i g'(\psi_i)}{g(\psi_i)} \right).$$

Finally, notice that $\frac{\psi_i g'(\psi_i)}{g(\psi_i)} = \frac{v'\left(n_i^f\right)}{n_i^f v''\left(n_i^f\right)} = \zeta_i$.

□

Theorem 2.

Proof. There are three cases we should consider. In the first one, the interior formal income schedule, as implied by conditions (25)-(27), is increasing in type. In such case, by Theorem 1, the interior allocation is optimal. Second, the interior income schedule may be decreasing. In this case, we will derive formally the necessary optimality condition (30). Finally, the interior formal income schedule may be locally flat. In the last part of this proof we will show that in this case interior optimality conditions (25)-(27) and the optimal kink formula (30) are equivalent.

If the interior income schedule is at some point decreasing, (25)-(27) are no longer necessary for the optimum - they lead to a violation of the incentive-compatibility. The optimal formal income schedule will involve bunching some types, and the planner may perform a perturbation of the formal income level at which the types are bunched. No such perturbation is available, if (30) holds. Now we will derive formally this optimality condition. The derivations follow the same steps as Doligalski [2015], we provide them here for the sake of completeness. Suppose that the formal income schedule y^f is constant at the segment of types $[a,c]$. Let's marginally decrease the formal income of types $[a,c)$. Since we don't change the allocation of types below a, we have to make sure that V_a is unchanged - otherwise the utility schedule would become discontinuous. Together with the cut of the formal income, we have to introduce a change in the total tax paid at this income level $dT_a = 1 + \frac{\partial V_a}{\partial y^f} = t_a$. Since all types $[a,c)$ are affected, the tax revenue loss is equal

$$t_a \left(M_c - M_a\right). \tag{46}$$

Although this perturbation does not affect the utility of type a, it does affect the utility of all other types at the kink. The utility impact of the perturbation of some type $i \in (a,c)$ equals $dU_i = 1 + \frac{\partial V_i}{\partial y^f} - dT_a = \frac{v'(n_a)}{\phi_a} - \frac{v'(n_i)}{\phi_i}$. The welfare loss due to this utility change is

$$\int_a^c \Delta MRS_i w_i d\mu, \tag{47}$$

where $\Delta MRS_i = \frac{v'(n_a)}{\phi_a} - \frac{v'(n_i)}{\phi_i}$ is the distance in terms of the marginal rate of substitution from a, the initial type at the kink. Having the fiscal and welfare loss at the kink, we can add them into a cost of increasing distortions at the kink, $D^K_{a,c}$. The definition (29) involves also a normalization term $\frac{1}{t_c - t_a,}$ which makes sure that the perturbation results in the unit change of the utility of type c.

The change of allocation at the kink results in a shift in the utility level of all types above the kink. Since we normalized the perturbation such that the utility level of type c falls by a unit, the benefit of perturbation is the familiar term $\int_c^1 N_j dj$, which captures the gains from non-distortionary taxation of types above (and including) c. By equalizing the cost and the benefit of the perturbation, we arrive at the optimality condition (30).

Finally, let's consider the case when the interior income schedule is flat on the segment $[a,c]$. The proof of equivalence of the interior optimality conditions and the optimality condition at the kink follows the same steps as the proof of Theorem 2 in Doligalski [2015], with one qualification. The proof involves integrating $\frac{\partial \dot{V}_i}{\partial y^f} = -\dot{MRS}_i$ over the segment $[a,c]$. As we demonstrated before, the expression for $\frac{\partial \dot{V}_i}{\partial y^f}$ of marginal workers in general depends on the direction of change of formal income. However, in this specific case, when the interior formal income is flat, the expression $\frac{\partial \dot{V}_i}{\partial y^f}$ is well defined for the marginal workers. By Lemma 2, formal income is constant for the marginal workers if $\frac{\dot{\phi}_i}{\phi_i} + \frac{\dot{\psi}_i}{\psi_i}\zeta_i = 0$. It is easy to see that it implies

$$\frac{\dot{\phi}_i}{\phi_i}\left(1 + \frac{1}{\zeta_i}\right)\frac{\psi_i}{\phi_i} = \left(\frac{\dot{\phi}_i}{\phi_i} - \frac{\dot{\psi}_i}{\psi_i}\right)\frac{\psi_i}{\phi_i}. \tag{48}$$

The left-hand side is equal to $\frac{\partial \dot{V}_i}{\partial y^f}$ evaluated as if i was a formal worker, while the right-hand side is $\frac{\partial \dot{V}_i}{\partial y^f}$ as if i was a shadow worker. The two expressions coincide, so they determine the derivative of the marginal information rent of a marginal worker with respect to formal income. Therefore, the integral of $\frac{\partial \dot{V}_i}{\partial y^f}$ over the set $[a,c]$ is well defined.

□

Lemma 3.

Proof. Suppose that the interior allocation is not incentive compatible. Let's first consider the case in which the function G in the social welfare is affine. In this case term $N_j = (1 - w_j)\,\mu_j$ does not depend on the allocation. Whenever there is a kink, it is optimal to decrease the formal income of some types below the kink For these types, the marginal tax rate is increased or stays constant. For the types above the kink, there is no reason to change the tax rate. Hence, the marginal tax rates in the optimum are either equal or higher than in the interior allocation, so the incentives to join the shadow economy are strengthened.

Consider the general case of G concave. Then the large changes of the marginal tax rates below a given type i affect N_j for $j > i$. When the interior allocation is not incentive compatible, the optimal allocation will involve different marginal tax rates for the mass of agents at the kink. It means that the terms N_j will be different for all types above the kink, which changes the tax rate trade-off for all types. The tax rates on the entire type space (apart from the top) will differ between the optimum and the interior allocation. However, as before, the tax rates will be always higher in the optimum. The incentive compatibility in the optimum prevents the planner from reducing the information rents of the top types as much as in the interior allocation. Incentives to increase tax rates are higher everywhere because the benefit $\int_i^1 N_j dj$ is higher for every i.

□

Lemma 4.

Proof. It is just an interior optimality condition for the shadow worker (27). By Lemma 3, all the shadow workers from the interior allocation are shadow workers in the optimum.

□

C Estimation results of the factor F_i and top earners Pareto distribution.

Here we present the variables included in the vector X_i and the parameter estimates of β and γ obtained from the specification given by (39).

Table (4) lists the variables included in X_i with its corresponding description and associated category. The parameter estimates are presented in Table (5).

Finally Table XX presents the estimate of the Pareto distribution for top earners. The table contains the scale parameter estimates and the quantiles of formal productivity of top earners.

Table 4: Variables included in X_i

Variable	Description	Values
Worker characteristics		
Gender	Dummy variable equal to 1 for women	0-1
Age	Age of the worker	16-90
Age^2	Age squared	
Ed years	Number of education years	0-26
Degree	Highest degree achieved	1-5
		1 - no degree
		5 - postgraduate degree
Y work	Number of months worked in the last year	1-12
Experience	Number of months worked in the last job	0-720
First job	Dummy for the first job (1 if it is the first job)	0-1
Production unit (firm) characteristics		
Sector Man	Dummy for the manufacturing sector	0-1
Sector Fin	Dummy for financial intermediation	0-1
Sector ret	Dummy for the sales and retailers sector	0-1
Big city	Dummy for a firm in one of the two largest cities	0-1
Size	Categories for the number of workers	1-9
		1 - One worker
		9 - More than 101 workers
Production unit (Type of job) characteristics		
Lib	Dummy for a liberal occupation	0-1
Admin	Dummy for an administrative task	0-1
Seller	Dummy for sellers and related	0-1
Services	Dummy for a service task (bartender ..)	0-1
Worker-firm relationship		
Union	Dummy for labor union affiliation (1 if yes)	0-1
Agency	Dummy for agency hiring (1 if yes)	0-1
Seniority	Number of months of the worker in the firm	0-720

Table 5: Estimation results

Parameter	Point estimate	std. error	t-statistic	95% conf. interval	
γ_0^f	6.859	0.033	211.9	6.89	7.02
$\gamma_0^s - \gamma_0^f$	0.102	0.032	-3.2	-0.16	-0.04
γ_1^s	0.682	0.037	12.6	0.648	0.716
β-Gender	-0.077	0.005	-11.6	-0.06	-0.04
β-Age	0.025	0.001	13.1	0.01	0.02
β-Age^2	0.000	0.000	-8.8	0.00	0.00
β-Ed years	0.037	0.002	15.4	0.02	0.03
β-Degree	0.156	0.005	21.1	0.10	0.12
β-Sector Man	-0.098	0.006	-11.9	-0.08	-0.06
β-Sector Fin	0.156	0.015	6.9	0.08	0.14
β-Sector ret	-0.150	0.006	-16.9	-0.11	-0.09
β-Big city	0.010	0.007	1.0	-0.01	0.02
β-Size	0.032	0.001	18.7	0.02	0.02
β-Union	0.126	0.010	8.3	0.07	0.11
β-Agency	-0.144	0.005	-18.3	-0.11	-0.09
β-Seniority	0.001	0.000	17.9	0.00	0.00
β-Y work	0.029	0.001	18.4	0.02	0.02
β-First job	-0.053	0.008	-4.7	-0.05	-0.02
β-Experience	0.000	0.000	5.3	0.00	0.00
β-Lib	0.074	0.013	3.9	0.03	0.08
β-Admin	-0.272	0.009	-19.9	-0.20	-0.17
β-Seller	-0.186	0.014	-9.2	-0.15	-0.10
β-Services	-0.267	0.009	-19.3	-0.20	-0.16

Table 6: Pareto distribution estimates

Parameter	Point estimate	std. error	z-statistic	95% conf. interval	
Shape parameter	1.81	0.0018	953.34	1.806	1.833

Quantile	Formal productivity ϕ_i
1%	1.006
5%	1.029
10%	1.060
20%	1.131
25%	1.173
30%	1.218
40%	1.327
50%	1.467
60%	1.659
70%	1.945
75%	2.152
80%	2.434
90%	3.569
95%	5.235
99%	12.737